Lesson 3: Functions

Definition:

- Let A and B be nonempty sets. A ***function*** from A to B is a rule that assigns to *every element* of A a *unique* element in B. We call A the ***domain***, and B the ***codomain***, of the function. If the function is called f, we write $f{:}A \to B$. Given $x \in A$, its associated element in B is called its ***image*** under f. In other words, a function is a relation from A to B with the condition that for every element in the domain, there exists a unique image in the codomain (this is really two conditions: existence of an image and uniqueness of an image). We denote it $f(x)$, which is pronounced as " f of x."
- A function is a special type of relation

The reason why we say a function is special relation, because all of its first components cannot be mapped more than once.

Example:

1. Identify whether the following relations are functions or not?
   a. $A = \{(1,2), (3,4), (5,4), (9,3)\}$
   b. B ={(0,3),(1,4),(2,5),(6,7),(0,8)}
   c. $R = \{(x, y){:}\, x \text{ is the age of } y\}$
   d. $R = \{(x, y){:}\, x \text{ is the circumfrence of circle } y\}$

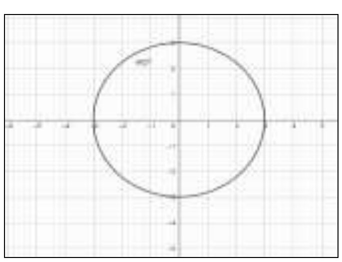
Solution:

a. A is a function since no first entry of A is mapped more than once
b. B is not a function, since 0 is mapped more than once.
c. R is not a function since there may have two or more y's with the same age.
d. R is a function since circumference of a circle is unique.

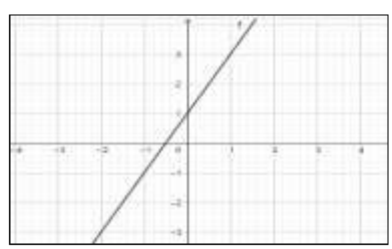
Vertical line test

A relation R is a function if and only if any vertical line intersects graph of R only once.

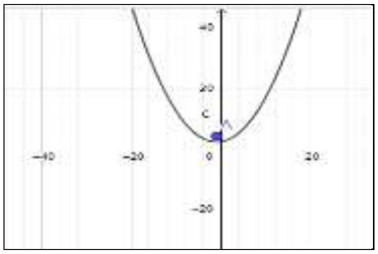
Examples: 1. Identify the following are function graphs or not?

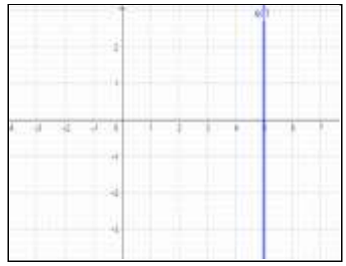


Not function



function





Function not function

2. Which of the following relations are functions?
   a. $R_1 = \{(x, y): x, y \in R \text{ and } y = x^2 - 3\}$
   b. $R_2 = \{(x, y): x, y \in Z \text{ and } y \leq x + 4\}$
   c. $R_3 = \{(x, y): x, y \in Q \text{ and } y = \sqrt{2 - x^2}$

Solution:

a. R_1 *is a function, since it is an up − ward paarabola*
b. R_2 is not a function, since it is a region.
c. R_3 *is* not a function, since it is a circle.

Note:
- Every function is a relation but every relation is not a function.
  - Every relation involving inequalities is not a function.

1.3.1. Types of functions

i. Power function
ii. Absolute value function
iii. Signnum function
iv. Greatest integer (floor) function

i. Power function

Definition: a function which can be written in the form of $f(x) = ax^r, where\ a, r \in R$

Some examples of power functions:

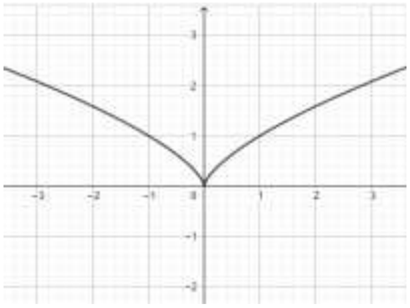
a. $f(x) = 3x^2$
b. $f(x) = x^{3.5}$
c. $g(x) = x^{\sqrt{2}}$

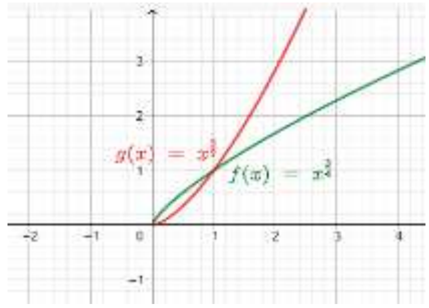
Note: $y = f(x) = ax^r$ *has variable base and fixed exponent.*

Graphs of some power functions:

1. A. $y = x^{\frac{m}{n}}$, m = even and n=odd, (m < n)
$Domain = \mathbb{R}$
Range = $[0, \infty)$

b. $y = x^{\frac{m}{n}}$, where m = odd and n = even,
Domain = $[0, \infty)$
Range = $[0, \infty)$





2. A. $y = x^{-\frac{1}{n}}$, n = odd

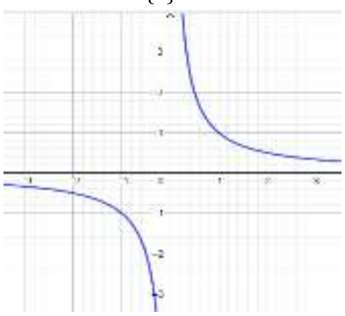
Domain = $\mathbb{R}\backslash_{\{0\}}$

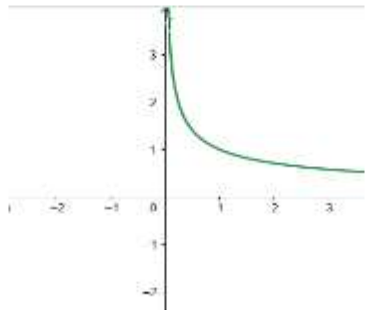
Range = $\mathbb{R}\backslash_{\{0\}}$

b. $y = x^{-\frac{1}{n}}$, n = even

Domain = $(0 , \infty)$

Range = $(0 , \infty)$





3. A. $y = x^{\frac{-m}{n}}$, m = even and n=odd

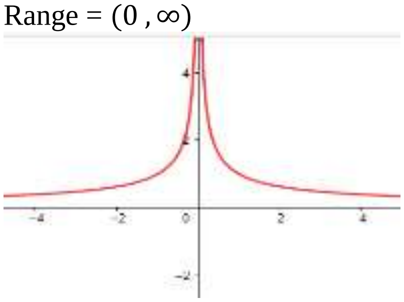
Domain = $\mathbb{R}\backslash_{\{0\}}$

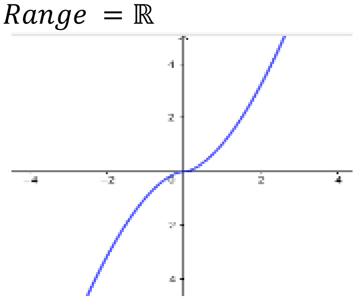
Range = $(0 , \infty)$

b. $y = x^{\frac{m}{n}}$, m = odd and n=odd (m > n)

$Domain = \mathbb{R}$

$Range = \mathbb{R}$





Examples:

1. Find the domain of each of the following power functions
   a. $f(x) = x^{1/3}$ b. $f(x) = x^{5/4}$ c. $f(x) = 5x^{-2/3}$ d. $f(x) = x^{-7/4}$
2. If $f(x) = ax^n, a \neq 0, and\ f(xy) = f(x)f(y), then\ what\ is\ the\ value\ of\ a?$
3. Show that $f(x) = 2x^n$ satisfies the property $f(xy) = f(x)f(y)$?

Solution:

1. a. $f(x) = x^{1/3}$ b. $f(x) = x^{5/4}$ c. $f(x) = 5x^{-2/3}$ d. $f(x) = x^{-7/4}$
   Domain = $\mathbb{R}$ domain = $[0 , \infty)$ domain = $\mathbb{R}\backslash_{\{0\}}$ domain = $(0 , \infty)$

2. $f(x) = ax^n$

.$a(xy)^n = ax^n\ ay^n$

.$ax^ny^n = a^2x^ny^n$

.$\frac{a^2}{a} = 1 \Rightarrow a = 1$

3. $f(x) = 2x^n$

. $2(xy)^n = 2x^n\ x\ 2y^n$

.$2x^ny^n = 4x^n\ y^n$

.$\Rightarrow 2 \neq 4$

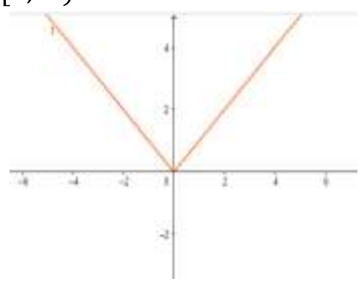
Therefore a = 1 therefore $f(x) = 2x^n$ doesnot satisfy $f(xy) = f(x)f(y)$

ii. Modulus (Absolute value) Function

Definition:

A real function $\boldsymbol{f: R \longrightarrow R}$ $\boldsymbol{defined\ by\ f(x) = |x| = \begin{cases} x, if\ x > 0 \\ 0, if\ x = 0 \\ -x, if\ x < 0 \end{cases}}$ $\boldsymbol{, for\ every\ x \in R}$ is the modulus function.

- Domain $f = R$
- Range $f = [0, \infty)$



Example:

1. Find domain and range of the following
   a. $f(x) = |x+1|$ b. $g(x) = \frac{1}{|x|}$ c. $h(x) = 1 - |x-2|$ d. $f(x) = \frac{1}{\sqrt{x+|x|}}$
2. Let $x = -2\ and\ y = 6, then$ find:
   a. $\left|2x + \frac{y}{6}\right|$ b. $|xy+1|$ c. $|x^2y|$ d. $\left|\frac{x}{y}\right|$

Solution:

1

a. $f(x) = |x+1| = \begin{cases} x+1, if x > -1 \\ 0, if\ x = -1 \\ -x-1, if\ x < -1 \end{cases}$

Domain = R

Range = $[-1, \infty)$

b. $g(x) = \frac{1}{|x|} = \begin{cases} \frac{1}{x}, if\ x > 0 \\ \nexists, if\ x = 0 \\ -\frac{1}{x}, if\ x < 0 \end{cases}$

Domain = $\mathbb{R}\backslash_{\{0\}}$

Range = $\mathbb{R}\backslash_{\{0\}}$

c. $h(x) = 1 - |x-2| = \begin{cases} 1 - x + 2 = 3 - x, if\ x > 2 \\ 1 - 0 = 1, if\ x = 2 \\ 1 - (-x+2) = x - 1, if\ x < 2 \end{cases}$

Domain = R

Range = R

d. $f(x) = \frac{1}{\sqrt{x+|x|}} = \begin{cases} \begin{cases} \frac{1}{\sqrt{2x}}, if\ x > 0 \\ \nexists\ \ if\ x \leq 0 \end{cases} \end{cases}$

Domain = $(0, \infty)$

Range = $(0, \infty)$

2. $x = -2$ and $y = 6$

a. $\left|2x + \frac{y}{6}\right| = \left|2(-2) + \frac{6}{6}\right| = 3$ b. $|xy + 1| = |-2(6) + 1| = 11$ c. $|x^2y| = |(-2)^2(6)| = 24$

d. $\left|\frac{x}{y}\right| = \left|-\frac{2}{6}\right| = \frac{1}{3}$

Properties:

For any real numbers x and y: $i. |x| = |-x|$ ii. $|xy| = |x||y|$

$iii. \left|\frac{x}{y}\right| = \frac{|x|}{|y|}$ $y \neq 0$, iv . $x \leq |x|$

$v.$ $|x + y| \leq |x| + |y|$ (triangular inequality)

Examples:

1. Find the solution set for the following equations
   a. $|x| = 8$ b. $|-5x + 7| = 2$ c. $|3x - 5| + 6 = 4$ d. $|3x + 5| + |2 - x| = 9$
2. Sketch the graph of the following functions
   a. $f(x) = |x| - x$ b. $f(x) = 3 - |x|$ c. $f(x) = |\sqrt{x}|$ d. $f(x) = \ln|x|$

Solution

1.

$a. |x| = 8:$ $case\ 1: if\ x > 0$ $then\ x = 8$ $case2: if\ x < 0$ $then - x = 8 \Rightarrow x = -8$

$\therefore \boldsymbol{s.s = \{-8, 8\}}$

$b. |-5x + 7| = 2$

$case\ 1:$ $if\ -5x + 7 \geq 0$

$x \leq \frac{7}{5}$

$-5x + 7 = 2$

$-5x = 2 - 7$

$-5x = -5$

$x = 1$

$case\ 2:$ $-5x + 7 < 0$

$x > \frac{7}{5}$

$-(-5x + 7) = 2 \implies 5x - 7 = 2$

$5x = 2 + 7$

$5x = 9$

$x = 9/5$

$\therefore \boldsymbol{s.s = \{1, 9/5\}}$

$c. |3x - 5| + 6 = 4$

Solution : it has no solution since, $|3x - 5| = 4 - 6 = -2$ $if and\ only\ if$ $|x| \geq 0$

d. $|3x+5|+|2-x|=9$

Solution

Case 1: $3x+5\geq 0 \; and \; 2-x\geq 0$

$x\geq -\frac{5}{3} \cap \; x\leq 2$

$x \in [-\frac{5}{3}, 2]$

$3x+5+2-x=9$

$2x+7=9$

$2x=9-7$

$2x=2$

$x=1$

case2: $3x+5\geq 0 \; and \; 2-x<0$

$x\geq -\frac{5}{3} \cap \; x>2$

$x>2$

$-(3x+5)-(2-x)=9$

$-3x-5-2+x=9$

$-2x-7=9$

$-2x=9+7$

$-2x=16$

$\boldsymbol{x=-8 \notin \{x: x>2\}}$

case3: $3x+5<0 \; and \; 2-x\geq 0$

$x<-\frac{5}{3} \; \cap \; x\leq 2$

$x<-\frac{5}{3}$

$-(3x+5)+2-x=9$

$-3x-5+2-x=9$

$-4x-3=9$

$-4x=9+3$

$-4x=12$

$x=-3$

Case4: $3x+5<0 \; and \; 2-x<0$

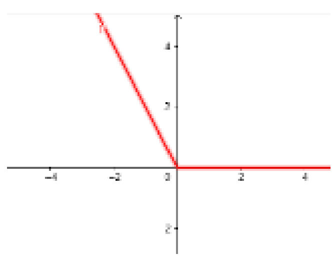
$x<-\frac{5}{3} \; \cap \; x>2$

$s.s.= \emptyset$

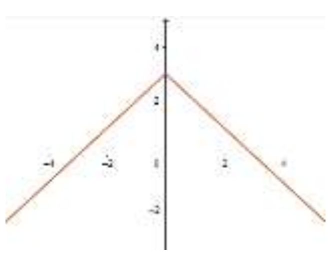
$\therefore \boldsymbol{s.s=\{-3, 1\}}$

1. **a.** $\boldsymbol{f(x)=x-|x|=\begin{cases} x-x=0, if \; x\geq 0 \\ -x-x=-2x, if \; x<0 \end{cases}}$

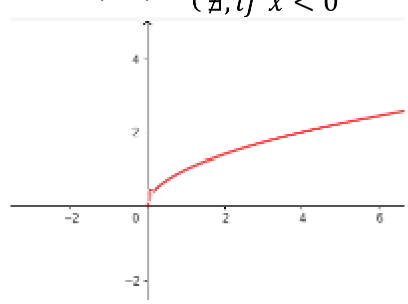
NB. The graph lies on the x- axis for $x\geq 0$ and a straight line $l: y=-2x \; for \; x<0$



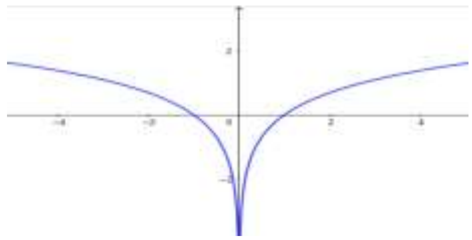
b. $f(x)=3-|x|=\begin{cases} 3-x, if \; x\geq 0 \\ 3+x, if \; x<0 \end{cases}$



c. $f(x)=|\sqrt{x}|=\begin{cases} \sqrt{x}, if x\geq 0 \\ \nexists, if \; x<0 \end{cases}$



c. $f(x)=\ln|x|=\begin{cases} lnx, if \; x\neq 0 \\ \nexists, if \;\; x= \;\; 0 \end{cases}.$



Absolutes Value Involving Inequalities

Take it as a snap:

- $|x| \leq a \iff -a \leq x \leq a$
- $|x| \geq a \iff x \leq -a \text{ or } x \geq a$

Example:

1. Solve the following inequalities

a. $|x| < 3$ b. $|2x+1| > 4$ c. $|2x-3|+5 \leq x-2$ d. $\left|\frac{2}{x}+5\right| \geq \frac{2}{3}$

Solution:

a. $|x| < 3$

Case1: $x < 3$ *and case* 2 : $-x < 3 \Rightarrow x > -3$

Take union of them

$\{x: x < 3\} \cup \{x: x > -3\}$

Therefore, $|x| < 3 \iff -3 < x < 3$

b. $|2x+1| > 4$

case1 $2x+1 > 4$ case 2: $2x+1 < -4$

$2x > 4-1$ $2x < -4-1$

$2x > 3$ $2x < -5$

$x > \frac{3}{2}$ $x < -\frac{5}{2}$

$\therefore$ $\boldsymbol{|2x+1| > 4 \iff x < -\frac{5}{2} \text{ or } x > \frac{3}{2}}$

c. $|2x-3|+5 \leq x-2$

Re-write as $|2x-3| \leq x-7$

Case1: $2x-3 \leq x-7$ case 2: $-(2x-3) \leq x-7$

$2x - x \leq -7+3$ $-2x+3 \leq x-7$

$x \leq -4$ $-2x - x \leq -7-3$

$-3x \leq -10$

$x \geq \frac{10}{3}$

$\therefore$ $|2x-3|+5 \leq x-2 \iff x \leq -4 \text{ or } x \geq \frac{10}{3}$

d. $\left|\frac{2}{x}+5\right| \geq \frac{2}{3}$

$\boldsymbol{case1}$: $\frac{2}{x}+5 \geq \frac{2}{3}$ $case2$: $\frac{2}{x}+5 \leq -\frac{2}{3}$

$\frac{2}{x} \geq \frac{2}{3}-5$ $\frac{2}{x} \leq -\frac{2}{3}-5$

$\frac{2}{x} \geq -\frac{13}{3}$ $\frac{2}{x} \leq -\frac{17}{3}$

$x \geq -\frac{6}{13}$ $x \leq -\frac{6}{17}$

$\therefore$ $\left|\frac{2}{x}+5\right| \geq \frac{2}{3} \iff x \leq -\frac{6}{17} \text{ or } x \geq -\frac{6}{13}$

iii. Signum Function

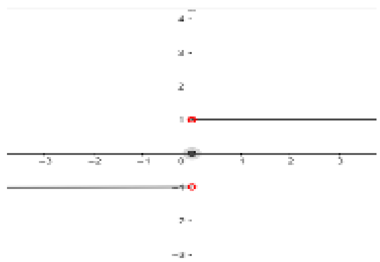
Definition:

The real function $f: R \longrightarrow R, defined\ by\ f(x) = sgnx = \begin{cases} 1, if\ x > 0 \\ 0\ , if x = 0 \\ -1\ , if\ x < 0 \end{cases}$ is the signum function.

Domain $f = R\ \ and\ \ Range\ \ f = \{1\ ,0\ ,1\}$

For ant real number $x, \frac{|x|}{x} = \begin{cases} \frac{x}{x} = 1\ for\ x > 0 \\ \nexists\ for\ \ x = 0 \\ \frac{x}{-x} = -1\ for\ x < 0 \end{cases}$ $\ \ implies\ sgnx = \begin{cases} \frac{|x|}{x} if\ x \neq 0 \\ 0\ \ \ if\ x = 0 \end{cases}$

Graph of $sgnx$



Examples:

1. Evaluate the following functions
   a. $sgn(-3.45)$ b. $6sgn\ \pi$ c. $sgn(0.000125485)$ d. $sign(e)$ e. $sgn\ (\frac{5}{2})$
2. Fin the solution set of the following functions
   a. $sgn(x+5) = -1$ b. $3 - sgn(2x+1) = 2$ c. $2sgn(3-7x) + 5 = 5$
3. Sketch the graph, and determine domain and range for the following functions
   a. f(x) = $x^2 sgnx$ $b.\ \ f(x) = xsgnx$ $c. f(x) = x - sgnx$ $d.\ \ |sgnx|$

solution:

1a. $sgn(-3.45) = 1$, since -3.45 < 0 b. $6sgn\ \pi = 6(ssgn\ 3.14) = 6 * 1 = 6$

c.$sgn(0.000125485) = 1, since\ 0.00012585 > 0$ d. $sign(e) = 1, since\ e = 2.718282 > 0$

e. $sgn\ \left(\frac{5}{2}\right) = 1\ , since \frac{5}{2} = 2.5 > 0$

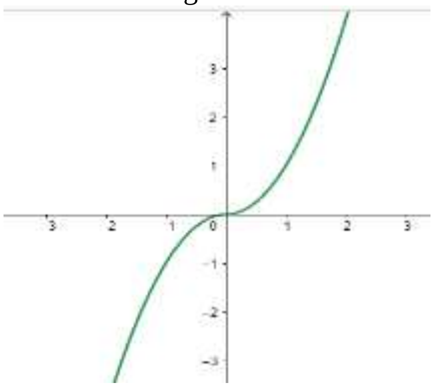
2. a. $sgn(x+5) = -1$
$x + 5 < 0$
$x < -5$
S. S= $\{x: x < -5\}$

b. $3 - sgn(2x+1) = 2$
$-sgn(2x+1) = 2 - 3$
$-sgn(2x+1) = -1$
$sgn(2x+1) = 1$
$.2x + 1 > 0$
$2x > -1$
$x > -\frac{1}{2}$
$s.s = \{x: x > -\frac{1}{2}\}$

c. $2sgn(3-7x) + 5 = 5$
$2sgn(3-7x) = 5 - 5$
$2sgn(3-7x) = 0$
$3 - 7x = 0$
$7x = 3$
$x = \frac{3}{7}$
$s.s = \left\{\frac{3}{7}\right\}$

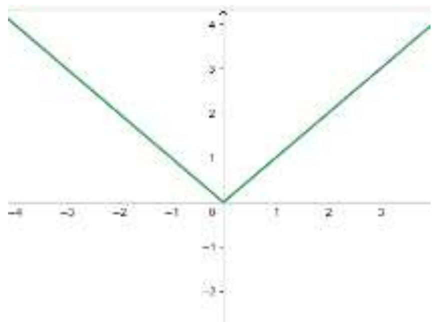
3.a. f(x) $= x^2 sgnx = \begin{cases} x^2\, , if\ x > 0 \\ 0, if\ x = 0 \\ -x^2, if\ x < 0 \end{cases}$

Domain = R
Range = R



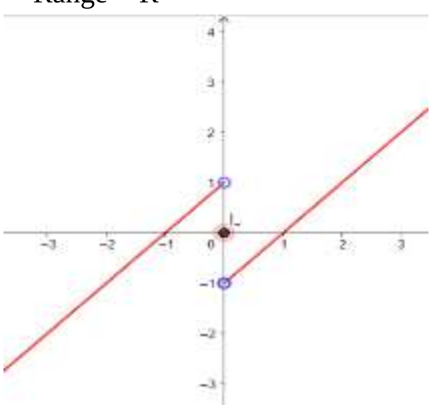
b. $f(x) = xsgnx = \begin{cases} x, if\ x > 0 \\ 0, if\ x = 0 \\ -x, if\ x < 0 \end{cases} = |x|$

Domain = R
Range $= [0\, , \infty)$



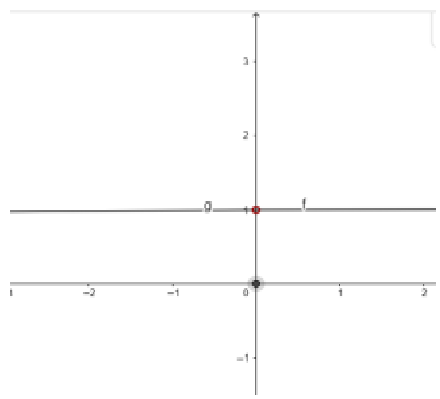
C. $f(x) = x - sgnx = \begin{cases} x - 1, if x > 0 \\ 0\, , if\ x = 0 \\ 1 - x, if\ x < 0 \end{cases}$

Domain = R
Range = R



d. $|sgnx| = \begin{cases} 1, if\ x \neq 0 \\ 0\, , if\ x = 0 \end{cases}$

Domain = R
Range = $\{0\, , 1\}$



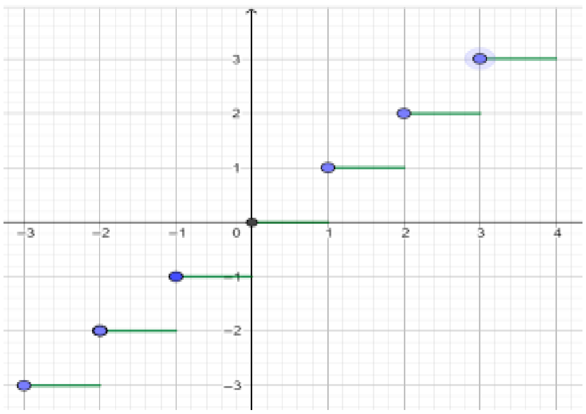
iv. Greatest Integer Function (Floor or Step Function)

Definition: The real function $f: R \longrightarrow R, defined\ by\ f(x) = \lfloor x \rfloor, x \in \boldsymbol{R}$ assumes the value of the greatest integer less than or equal to x is called the **greatest integer function**.

❖ Domain $f = R$ and Range $f = \mathbb{Z}$

Graph of $f(x) = \lfloor x \rfloor$

x	[-3,-2)	[-2,-1)	[-1,0)	[0,1)	[1,2)	[2,3)	[3,4)	[4,5)
$f(x) = \lfloor x \rfloor$	**-3**	**-2**	**-1**	**0**	**1**	**2**	**3**	**4**



Examples:

1. Evaluate a. $\lfloor -1.002 \rfloor$ **b.** $\lfloor 3.145 \rfloor$ **c.** $\lfloor 0 \rfloor$ **d.** $\lfloor 50 \rfloor$ **e.** $\lfloor \pi \rfloor$ **f.** $\lfloor -\sqrt{3} \rfloor$ **g.** $\left\lfloor \frac{19}{20} \right\rfloor$

Solution:

a. Greatest integer less than or equal to -1.002 **is -2.**
b. Greatest integer less than or equal to 3.145 is **3**
c. Greatest integer less than or equal to 0 is 0
d. Greatest integer less than or equal to 50 is 50.
e. Greatest integer less than or equal to $\pi = 3.14214 \ldots$ *is* **3**
f. $-2 < -\sqrt{3} < -1$, the greatest integer less than or equal to $-\sqrt{3}$ is -2.
g. $\frac{19}{20} = 0.95$, *then the greatest integer less than or equal to* 0.95 *is* 0.

NB $\lfloor x \rfloor = \begin{cases} x, & if \quad x = \mathbb{Z} \\ \mathbb{Z} \leq x, & if\ x \neq \mathbb{Z} \end{cases}$

Floor Functions Involving Equations

Note: To solve equations involving floor function we can use the description:

$\lfloor x \rfloor = a \iff a \leq x < a + 1$, *where* $a \in \mathbb{Z}$ *and* $x \in R$

Examples:

1. Solve the following

   a. $\lfloor \boldsymbol{x} \rfloor = -5$ **b.** $\lfloor 2x+3 \rfloor = 9$ **c.** $\left\lfloor 5-\frac{1}{2x} \right\rfloor = 0$ **d.** $\lfloor 1+\lfloor x \rfloor \rfloor = 6$

   e. $\lfloor x \rfloor^2 + \lfloor x \rfloor = 6$ **f.** $2^{\lfloor 2x \rfloor} = 8$ **g.** $\log_2 \lfloor x^2 \rfloor = 4$

2. Verify that $f(x)+f(y) \le f(x+y) \le x+y \; using \; f(x) = \lfloor x \rfloor \; when$

   a. x = 4.25, y = 6.32 b. x = -2.01 , y = $\boldsymbol{\pi}$

3. Sketch the graph of the following

   a. $f(x) = \lfloor 3x \rfloor$ b. $f(x) = 3\lfloor x \rfloor$ c. $f(x) = \lfloor 3+x \rfloor$

Solution:

1. a. $\lfloor x \rfloor = -5 \Rightarrow -5 \le x < -4$

b . $\lfloor 2x+3 \rfloor = 9 \Rightarrow 9 \le 2x+3 < 9+1$

$.\Rightarrow 9-3 \le 2x < 9+4$

$\Rightarrow 6 \le 2x < 13$

$\Rightarrow 3 \le x < \frac{13}{2}$

c. $\left\lfloor 5-\frac{1}{2x} \right\rfloor \Rightarrow \left\lfloor \frac{10x-1}{2x} \right\rfloor = 0$

$\Rightarrow 0 \le 10x-1 < 1$

$\Rightarrow 1 \le 10x < 2$

$\Rightarrow \frac{1}{10} \le x < \frac{1}{5}$

d. $\lfloor 1+\lfloor x \rfloor \rfloor = 6, \Rightarrow 6 \le 1+\lfloor x \rfloor < 6+1$

$\Rightarrow 5 \le \lfloor x \rfloor < 6$

$\Rightarrow \lfloor x \rfloor = Z = 5 \Leftrightarrow 5 \le x < 6$

e. $\lfloor x \rfloor^2 + \lfloor x \rfloor = 6 \Rightarrow \lfloor x \rfloor^2 + \lfloor x \rfloor - 6 = 0$

$\Rightarrow \lfloor x \rfloor + 3)(\lfloor x \rfloor - 2) = 0$

$\Rightarrow \lfloor x \rfloor + 3 = 0 \; or \; \lfloor x \rfloor - 2 = 0$

$\Rightarrow \lfloor x \rfloor = -3 \; or \; \lfloor x \rfloor = 2$

$.-3 \le x < -3+1 \; or \; 2 \le x < 2+1$

$-3 \le x < -2 \quad or \; 2 \le x < 3$

f. $2^{\lfloor 2x \rfloor} = 8 \Rightarrow 2^{\lfloor 2x \rfloor} = 2^3$

$\lfloor 2x \rfloor = 3$

$3 \le 2x < 3+1$

$\frac{3}{2} \le x < 2$

g. $\log_2 \lfloor x^2 \rfloor = 4$

$\log_2 \lfloor x^2 \rfloor = 4$

$\lfloor x^2 \rfloor = 16 \Leftrightarrow 16 \le x^2 < 17$

$4 \le \; |x| < \sqrt{17}$

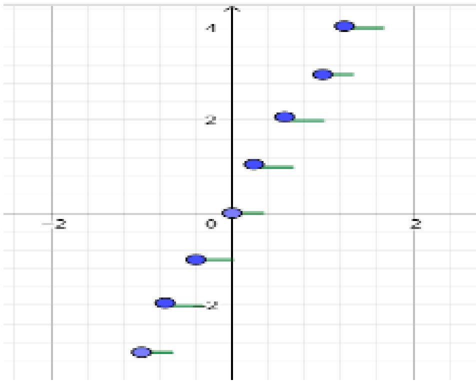
2. $f(x)+f(y) \le f(x+y) \le x+y$

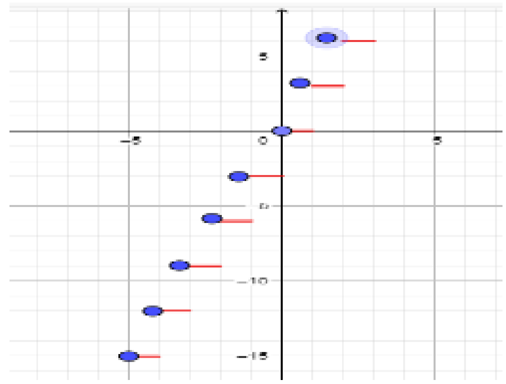
   a) $\lfloor 4.25 \rfloor + \lfloor 6.32 \rfloor \le \lfloor 4.25+6.32 \rfloor \le 4.25+6.32$

   $.4+6 \le 10 \le 10.57 \quad \Rightarrow 10 \le 10 \le 10.57$

   b) Left as an exercise for you!!!!!

3. $a. f(x) = \lfloor 3x \rfloor$ b. $f(x) = 3\lfloor x \rfloor$





c) c $f(x) = \lfloor 3 + x \rfloor$