Lesson 3: Rational functions and their graphs

Definition:

A function f which is written of the form $f(x) = \frac{P(x)}{Q(x)}$ where $P(x) and\ Q(x)$ are polynomials, $Q(x) \neq 0$ is a rational function.

- **Domain of** $\boldsymbol{f(x) = \frac{P(x)}{Q(x)} = \{x \in \mathbb{R} : Q(x) \neq 0\}}$

Example:

1. Find $f(0), f(1) and\ f(2) for\ the\ function\ f(x) = \frac{5x-8}{2-3x}$?
2. Determine the domain of $f(x) = \frac{5-3x}{1-x+2x^2-2x^3}$
3. If the ordered pair $(1,3) and\ (0,6)$ belongs to the function $f(x) = \frac{ax^3-2bx+3}{ax^2+b}$, then what is the value of a and b?
4. Find the number or the numbers that makes the denominator zero?

   a. $f(x) = \frac{x^2+1}{x^2-1}$ b. $f(x) = \frac{x^2-x-12}{x^2+x-6}$

Solution:

1. $f(x) = \frac{5x-8}{2-3x}$ $\Rightarrow f(0) = \frac{5(0)-6}{2-3(0)} = -3$

   $\Rightarrow f(1) = \frac{5(1)-6}{2-3(1)} = 1$

   $\Rightarrow f(1) = \frac{5(2)-6}{2-3(2)} = \frac{4}{-4} = -1$

2. domain of $f(x) = \frac{5-3x}{1-x+2x^2-2x^3}$

   steps to find: 1. Factorize the denominator

   2.find the zeros of denominator

   $\boldsymbol{Q(x) = 0} \Rightarrow 1 - x + 2x^2 - 2x^3 = 0 \quad \Leftrightarrow 1 - x + 2x^2(1-x) = 0$

   $\Leftrightarrow 1 - x + 2x^2(1-x) = 0$

   $\Leftrightarrow (1-x)(1+2x^2) = 0$

   $\Leftrightarrow (1-x)(1+2x^2) = 0 \Rightarrow x = 1\ and\ 2x^2 + 1 \neq 0$

   Domain = $R/\{1\}$

3. $f(x) = \frac{ax^3-2bx+3}{ax^2+b}$ $and\ f(1) = 3, f(0) = 6$

   $\Rightarrow f(0),\ \frac{a(0)-2b(0)+3}{a(0)+b} = 6 \Rightarrow (\frac{3}{b} = 6 \Rightarrow b = \frac{1}{2}$

   $\Rightarrow f(1), \frac{a(1)-2b(1)+3}{a(1)+b} = \frac{a-2b+3}{a+b} = 3 \Rightarrow a - 2\left(\frac{1}{2}\right) + 3 = 3\left(a + \frac{1}{2}\right)$

   $\Rightarrow a + 2 = 3a + \frac{3}{2} \Rightarrow 2a = 2 - \frac{3}{2} \Rightarrow a = \frac{1}{4}$

   $\therefore\ \boldsymbol{(a, b) = (\frac{1}{4}, \frac{1}{2})}$

4. a. $f(x) = \frac{x^2+1}{x^2-1} \Rightarrow x^2 - 1 = 0 \Rightarrow x = \pm 1$

the numbers that make denominator zero are -1 and 1

b. $f(x) = \frac{x^2-x-12}{x^2+x-6} \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x+3)(x-2) = 0 \Rightarrow x = -3 \; or \; x = 2$

1.4. Graph of Rational Functions

Procedures following to draw graph of $f(x) = \frac{N(x)}{D(x)}$, where $N(x) = numerator, D(x) =$ denominator and are polynomials.

- ✓ Determine domain
- ✓ Find x and y intercepts if any
- ✓ Identify asymptotes
- ✓ Check holes if any
- ✓ Study behavior of the graph near asymptotes
- ✓ Sketch the graph

Let $f(x) = \frac{N(x)}{D(x)}$ be a rational function:-

- domain = $R/\{D(x) \neq 0\}$
- x- intercept, at y = 0 and y- intercept at x = 0

1.4.1. Asymptotes

- An asymptote of a curve is a line such that the distance between the curve and the line approaches to zero as one or both of x or y coordinates tends to infinity. Or viceversal.

Types of asymptotes

i vertical asymptote

ii horizontal asymptotes

iii Oblique(slant) asymptote

a) Vertical Asymptotes

- Vertical line *l* obtained by $D(x) = 0$

Example:

1. Find the vertical asymptotes

a. $f(x) = \frac{2x+1}{x^2-x-2}$ b. $f(x) = \frac{x^2+5x+6}{x^2-9}$

Solution:

a. $f(x) = \frac{2x+1}{x^2-x-2}$

$,\Rightarrow D(x) = 0$

$,\Rightarrow x^2 - x - 2 = 0$

$,\Rightarrow (x-2)(x+1) = 0$

$,\Rightarrow x = 2 \; or \; x = -1$

Therefore vertical asymptotes at

$,x = 2 \; and \; x = -1$

b. $f(x) = \frac{x^2+5x+6}{x^2-9} = \frac{(x+2)(x+3)}{(x-3)(x+3)} = \frac{x+2}{x-3}$

$,\Rightarrow D(x) = 0$

$\Rightarrow x - 3 = 0$

$\Rightarrow x = 3$

vertical asymptotes at x =3

c. $f(x) = \frac{x+2}{x}$

$, \Longrightarrow D(x) = 0$

x = 0

Vertical asymptote at x = 0

ii. Horizontal Asymptotes

- $Degree\ N(x) < Degree\ D(x),$ the horizontal asymptote is always at$l: y = 0$
- $Degree\ N(x) = Degree\ D(x),$ the horizontal asymptote is$at\ l: y = numerical\ quotient$

Example:

2. Find horizontal asymptotes

   a. $f(x) = \frac{5x}{x^2+1}$ b. $f(x) = \frac{5x^2+2x-4}{2x^2-6x+1}$

Solution:

a. $f(x) = \frac{5x}{x^2+1}$

$,degreee\ of\ N(x) < degree\ of D(x)$

$,y = o\ is\ the\ horizontal\ asymptote$

b. $f(x) = \frac{5x^2+2x-4}{2x^2-6x+1}$

$degree\ \ of\ N(x) = degree\ of D(x)$

horizontal asymptote at $y = \frac{5x^2}{2x^2} = \frac{5}{2}$

i. Oblique asymptote

- Degree of $N(x) > D(x)\ by\ one\ , then\ the\ function\ has\ oblique\ asymptote$
- It must be determined by long division
- If a function has an oblique asymptote, then it doesn't have horizontal asymptote.

Example: determine the oblique asymptotes

a. $f(x) = \frac{3x^2+5}{x-6}$ b.$f(x) = \frac{6x^3+2}{3x^2+1}$ c. $g(x) = \frac{5x^3+3}{12x-7}$

Solution:

a. $f(x) = \frac{3x^2+5}{x-6} = 3x + 18 + \frac{113}{x-6}$

, then its oblique asymptote is at$y = 3x + 18$

b. $f(x) = \frac{6x^3+2}{3x^2+1} = 2x - 1 + \frac{3}{3x^2+1}$

Then oblique asymptote at $y = 2x - 1$

c. $g(x) = \frac{5x^3+3}{12x-7}$

degree N(x) is two more than degree of D(x)

then it has no oblique asymptote

Hole of the graph

- It is detected when factors $of\ N(x)\ and\ D(x)\ are$ cancelled to each other.

Example 3 :

Identify whether the function has hole or not

1. $f(x) = \frac{x^2-5x-6}{x^2-1}$ 2. $f(x) = \frac{3x+1}{3x^2+4x+1}$

Solution: 1. $f(x) = \frac{x^2-5x-6}{x^2-1} = \frac{(x+1)(x-6)}{(x-1)(x+1)} = \frac{x-6}{x-1}$

$f\ has\ hole\ at\ (x, y) = \left(-1\ , \frac{-1-6}{-1-1}\right) = (-1, \frac{7}{2}\ )$, because it $x + 1$ is cancelled from both the numerator and denominator

2. $f(x) = \frac{3x+1}{3x^2+4x+1} = \frac{3x+1}{(3x+1)(x+1)} = \frac{1}{x+1}$, $3x+1$ is the cancelled factor from both of numerator and denominator. F has hole at $(x, y) = \left(-\frac{1}{3}, \frac{1}{-\frac{1}{3}+1}\right) = \left(-\frac{1}{3}, \frac{3}{2}\right)$

Graph sketching

1. For each of the following functions find:
   i. Domain
   ii. x- and y- intercepts
   iii. asymptotes
   iv. holes if any
   v. and draw the graph

a. $f(x) = \frac{x+1}{x^2-1}$ b. $f(x) = \frac{x^2-x-6}{x-2}$ c. $f(x) = \frac{x^3-1}{x^2-1}$ d. $f(x) = \frac{x+1}{x-3}$
e. $f(x) = \frac{x^2-1}{x+1}$ f. $f(x) = \frac{x^2}{x^2+1}$ g. $f(x) = \frac{x^3+1}{x^2+6x+9}$

2. Show that the graph of $f(x) = \frac{3x^2+5x+1}{x^2-1}$ crosses its horizontal asymptote?

3. Show that the graph of $f(x) = \frac{x^3+x-1}{x^2}$ crosses its oblique asymptote?

Solution :

a. $f(x) = \frac{x+1}{x^2-1}$

i. domain: $\boldsymbol{x^2 - 1 = 0}$

,$\Longrightarrow (x-1)(x+1) = 0 \Longrightarrow x = \pm 1$

Dom $f = R\backslash\{-1,1\}$

ii. intercepts:

x- intercept when y=0,

,$\Leftrightarrow 0 = \frac{x+1}{x^2-1} = \frac{x+1}{(x+1)(x-1)} = \frac{1}{x-1} \Longrightarrow 1 = 0$

Is false it has no x intercept.

y – intercept when x = 0

$\Leftrightarrow y = \frac{0+1}{0-1} \Longrightarrow y = -1$

y – intercept at $(0, -1)$

iii. asymptotes

vertical asymptote

, $f(x) = \frac{x+1}{x^2-1} = \frac{x+1}{(x-1)(x+1)} = \frac{1}{x-1}$

, $D(x) = 0 \Longrightarrow x - 1 = 0 \Longrightarrow x = 1$

Vertical asymptote at $x = 1$

horizontal asymptote or oblique

degree N(x) < degree D (x)

horizontal asymptote at y = 0

No oblique asymptote!!

iv. Hole

, $f(x) = \frac{x+1}{x^2-1} = \frac{x+1}{(x-1)(x+1)}$

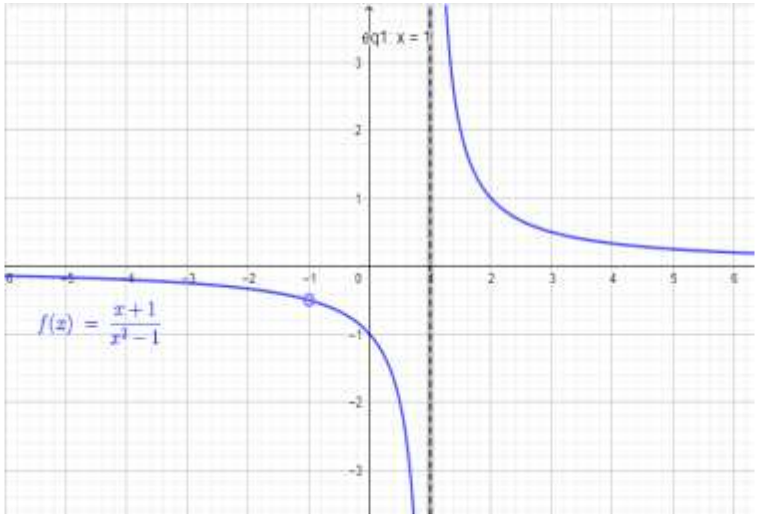
, x+1 is the cancelled factor

Hole at $(-1, -\frac{1}{2})$

v. Behavior of the graph

a. As x is getting closer and closer to vertical asymptote, y is increasing or decreasing indefinitely i.e. $x \longrightarrow 1^+, Y \longrightarrow \infty$ *and* $x \rightarrow 1^-, y \rightarrow -\infty$

b. As $x \rightarrow \infty, y \rightarrow H.A = 0$ *and* $x \rightarrow -\infty, y \rightarrow H.A$



b. $f(x) = \frac{x^2-x-6}{x-2}$

i. domain

, $x - 2 = 0 \Longrightarrow x = 2$

, $domf = \mathbb{R}\backslash\{2\}$

Ii. intercepts

x- intercepts when y = 0

, $\Longrightarrow 0 = \frac{x^2-x-6}{x-2} \Leftrightarrow x^2 - x - 6 = 0$

,$(x-3)(x+2) = 0 \Rightarrow x = 3, -2$

X – intercept at $(-2\,,0) and\ (\,3\,,0)$

y – intercept when x = 0

$y = \frac{0^2-0-6}{0-2} = 3$

y – intercept at $(0\,,3)$

i. Asymptotes

Vertical

$\mathrm{D}(x) = 0 \Leftrightarrow x - 2 = 0$

, $vertical\ asymptote\ at\ x = 2$

No horizontal asymptote!!

since $degN(x)\ >\ deg\ D(x)$

Oblique Asymptote

using long division

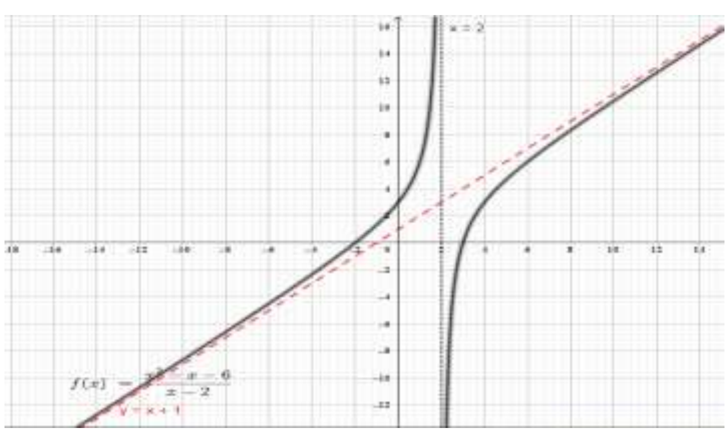
$f(x) = \frac{x^2-x-6}{x-2} = x + 1 - \frac{4}{x-2}$

oblique asymptote at $l\!: y = x + 1$

ii. Behavior of the graph

As $x \to 2^-, y \to \infty\ and\ x \to 2^+, y \to\ -\infty$

As $x \to \pm\infty, y \to l\!: y = x + 1$



A. f
B.
C. $f(x)=\frac{x^3-1}{x^2-1}=\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}$

, **i. domain**:

,$(x-1)(x+1)=0 \implies x=-1,1$

, $Dom\ f=\mathbb{R}\backslash\{-1,1\}$

ii. intercepts

x- intercept when y = 0

, $0=\frac{x^3-1}{x^2-1}=\frac{x^2+x+1}{x+1}$ $implies\ x^2+x+1\ cannot\ be\ zero$

It has no x-intercept

y- intercept when x = 0

$y=\frac{0^3-1}{0^2-1}\implies y=1$

y – intercept at (0 , 1)

i. Asymptotes

Vertical

,$f\ (x)=\frac{x^3-1}{x^2-1}=\frac{(x-1)(x^2+x+1)}{(x-1)(x+1)}=\frac{x^2+x+1}{x+1}$

, D(x) = 0$\implies x+1=0 \Rightarrow x=-1$

Vertical asymptote at $x=1$

oblique asymptote

$f(x)=\frac{x^2+x+1}{x+1}=x$

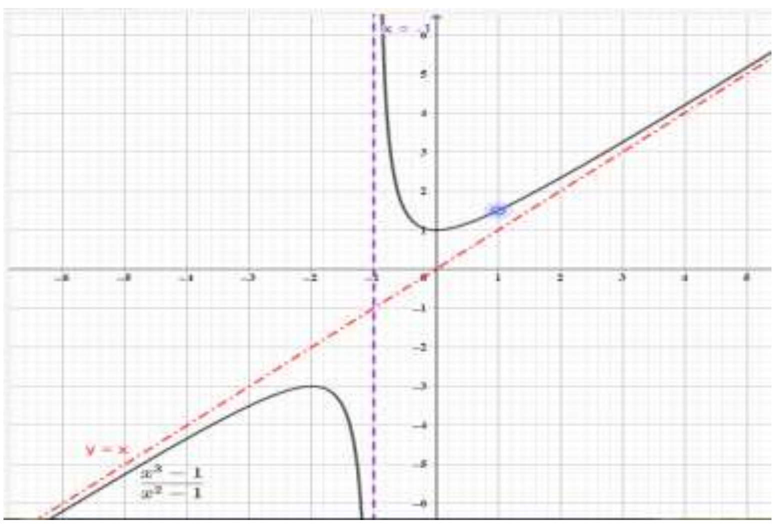
oblique asymptote at y = x

ii. Hole at (1, 3/2)

iii. Behavior of the graph

As $x\longrightarrow -1^+, y\longrightarrow\infty\ and\ x\rightarrow -1^-, y\rightarrow -\infty$

As $x\rightarrow\pm\infty\ y\rightarrow O.A.l{:}\ y=x$



d. $f(x)=\frac{x+1}{x-3}$

i. **domain :** $\boldsymbol{x}-3=0\Rightarrow x=3$

Domain $f=\mathbb{R}\backslash\{3\}$

ii.intercepts

x- Intercept when y =0

$0=\frac{x+1}{x-3}\implies x+1=0$

, $\implies x=-1$

X- Intercept at (-1, 0)

Y- intercept when x =0

$y=\frac{0+1}{0-3}\implies y=-\frac{1}{3}$

y – intercept at $(0, -\frac{1}{3})$

iv . Asymptotes

Vertical

, $f(x) = \frac{x+1}{x-3}$

, $\mathrm{D}(x) = 0 \Leftrightarrow x - 3 = 0$

,$x = 3$

vertical asymptote at$x = 3$

horizontal asymptote

$degN(x) = degD(x) = 1$

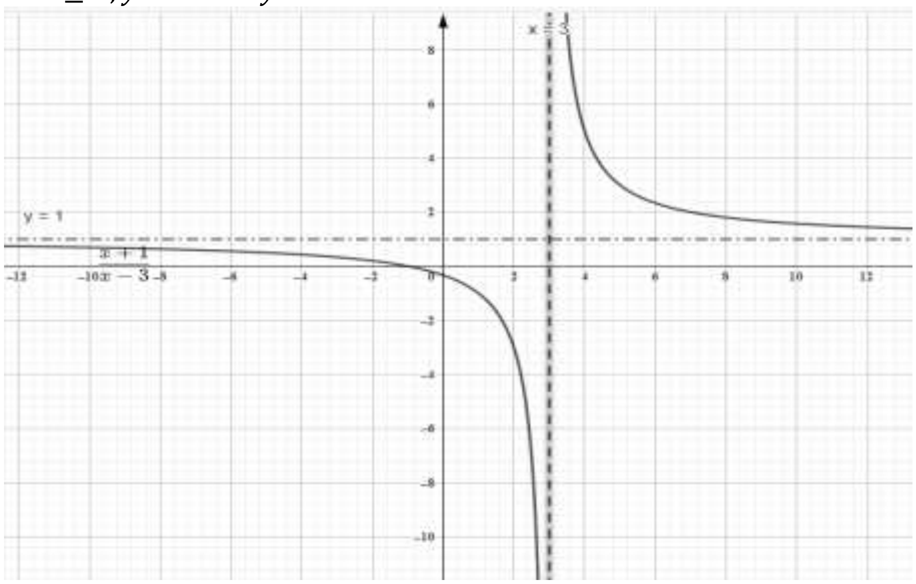
$H.A. at\ y = 1$

No Oblique Asymptote And Hole

iii. behavior of the graph

as $x \longrightarrow 3^{+}, y \longrightarrow \infty\ and\ x \rightarrow 3^{-}, y \rightarrow -\infty$

, as $x \rightarrow \pm\infty, y \rightarrow H.A\ l: y = 1$



e. $f(x) = \frac{x^2-1}{x+1}$

i. domain

,$x + 1 = 0 \implies x = -1$

$domf = R\{-1\}$

ii.vertical

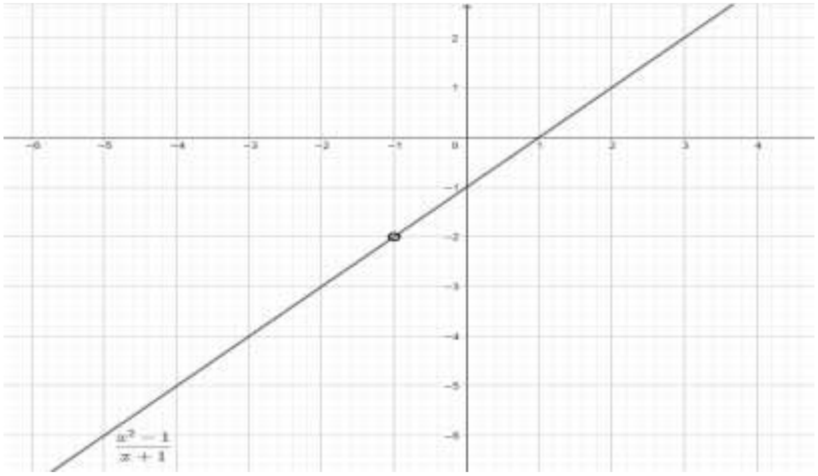
lets observe $f(x) = \frac{x^2-1}{x+1} = \frac{(x-1)(x+1)}{x+1} = x - 1$

it shows , there is no V.A.

Oblique Asymptote

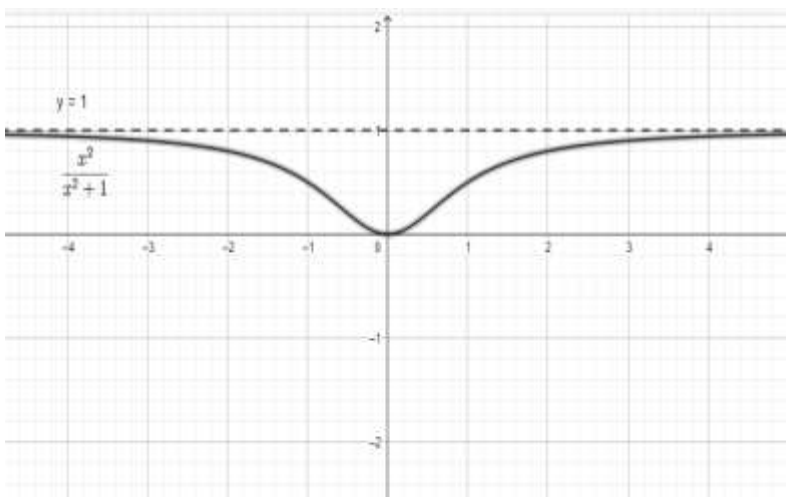
$y = x - 1\ is\ the\ O.A$

Hole at$(-1 , -2)$



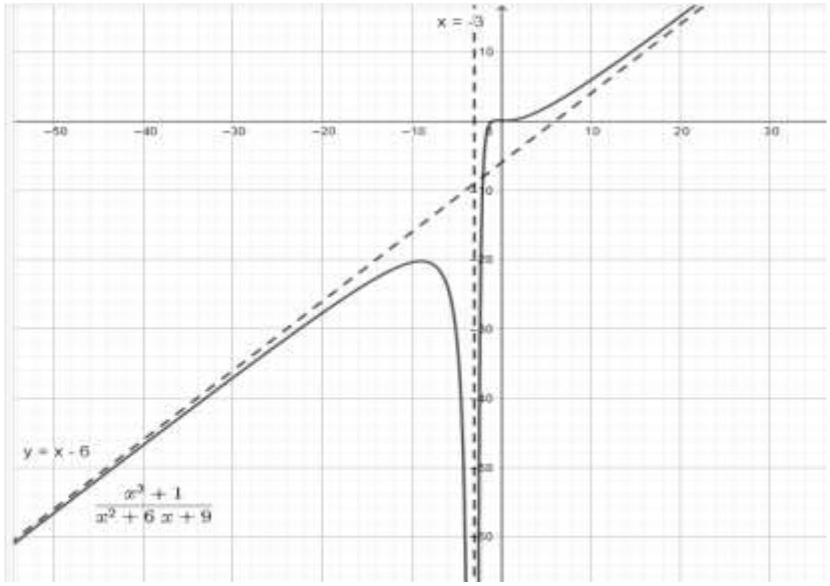
a. $f(x) = \frac{x^2}{x^2+1}$

- Domain f= R
- H.A. at y = 1
- No V.A. , NO HOLE
- X- AND Y- intercepr at (0 ,0)
-



g. $f(x) = \frac{x^3+1}{x^2+6x+9}$

- Domain$f = R\{-3\}$
- X- and y- intercepts at (-1 , 0) and (0 , 1/9)
- V.A. at $x = -3$
- O.A. at $y = x - 6$
- Behavior of the graph, $as\ x \to -3^+, y \to -\infty\ and\ x \to -3^-, y \to -\infty$



2. Show that the graph of $f(x) = \frac{3x^2+5x+1}{x^2-1}$ crosses its horizontal asymptote?

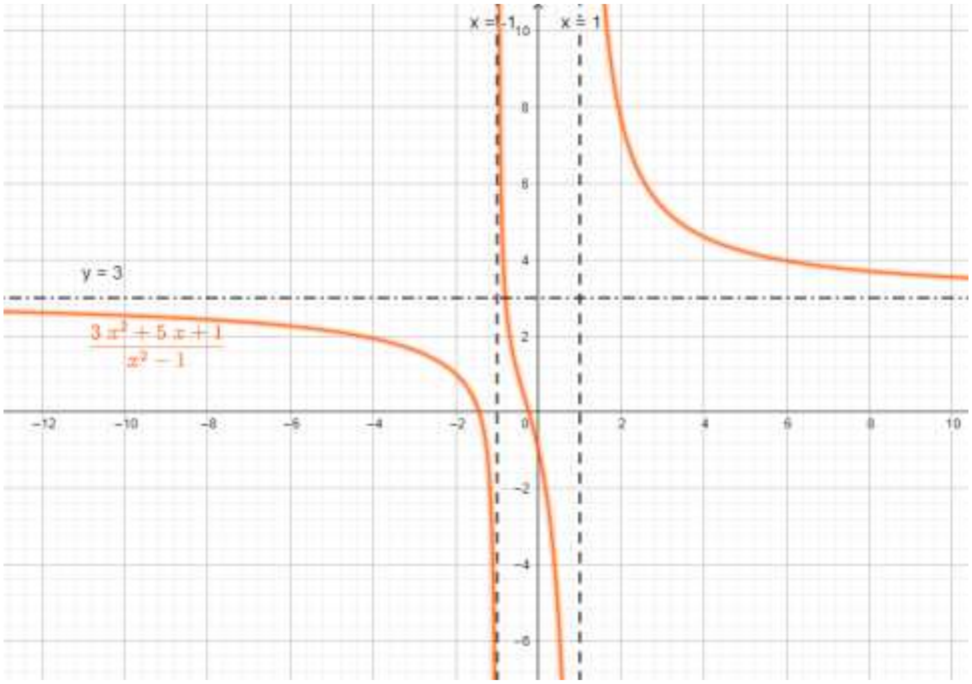
Answer :

$, f(x) = \frac{3x^2+5x+1}{x^2-1}$

- First we determine the horizontal asymptote:
- H.A: at $y = 3$
- Equate f with H.A. if it crosses
- $H.A = \frac{3x^2+5x+1}{x^2-1} \Longrightarrow 3 = \frac{3x^2+5x+1}{x^2-1}$
- $3x^2 - 3 = 3x^2 + 5x + 1 \Longrightarrow -3 = 5x + 1$
- $\Longrightarrow 5x = -4 \Longrightarrow x = -\frac{4}{5}$

Therefore the graph crosses its horizontal asymptote at $(-\frac{4}{5}, 3)$

Look at this on the graph below



3. Show that the graph of $f(x) = \frac{x^3+x-1}{x^2}$ crosses its oblique asymptote?

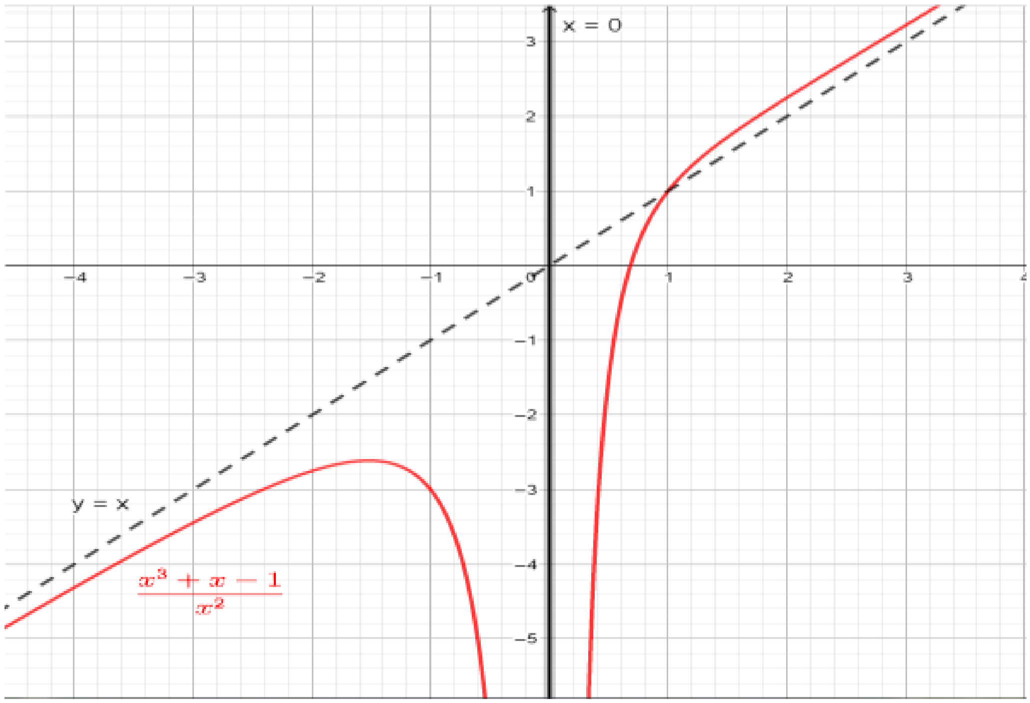
Solution :

Determine its oblique asymptote

- Deg. $N(x) > \deg D(x)$ by one
- use long division
- oblique asymptote is at $y = x$
- if it crosses Q.A. $F(x) = O.A.$
- Equate $x = \frac{x^3+x-1}{x^2} \Leftrightarrow x^3 = x^3 + x - 1$
- $0 = x - 1 \Longrightarrow x = 1$

The graph crosses its oblique asymptote at $(1\ ,1)$

See the graph below



Note:
1. The graph of any rational function never crosses its vertical asymptote.
2. The graph of any rational function may or may not cross its horizontal and oblique asymptotes.

Lesson 4: Application of Rational Expressions and Rational Functions

The concept is useful on our day today activities like:

- Work rate problems
- Comparison (Variation) problems
- Uniform motion problems
- Revenue problems
- Critical analysis and synthesis

Work rate and shared problems

Examples:

1. Roman can do her home-work assignments in 100 minutes. It takes Dergu about two hours to complete a given assignment. How long will it take two of them working together to complete the assignment?
2. Zerihun and Aschenaki painted a fence in four hours. If Ashenaki has painted the same fence before by himself in seven hours, how long would it take Zerihun on his own?
3. Zemenu and selam working together can do ajob in 6 hours. Zemenu working alone could have completed the job in 5 hours earlier than selam could have done alone. How much time could each worker need to complete the whole job alone?

General illustration to solve such kind of problems

Let A be the time taken to complete a certain job by worker one and B be the time taken by the second worker to complete the job.

Rate of accomplishing the job of the first worker $= \frac{1}{A}$ and rate of second worker $= \frac{1}{B}$

Rate of working together $\frac{1}{A} + \frac{1}{B} = \frac{1}{x}$, if x is the time it takes to complete the job together.

Solution:

1. Given Roman's rate of doing her assignment= $\frac{1}{100}$ in a minute

   Dergu's rate of doing the same assignment $= \frac{1}{120}$ in a minute

   And let x be the time taken in minute to complete the assignment when they do together

   Then $\frac{1}{100} + \frac{1}{120} = \frac{1}{x}$ is the general equation

   Implies $\frac{1}{100} + \frac{1}{120} = \frac{1}{x} \Longrightarrow \frac{120+100}{12000} = \frac{1}{x} \Leftrightarrow 220x = 12000$

   Therefore $x = \frac{12000}{220} = 54.54\ mimnutes$

Ans.:- Roman and Dergu take 54.54 minutes to complete their assignment together.

NB. You have to check that the joint work rate time is less than individuals work rate!!!!!