: Rational Equations And Rational Inequalities

1.3.1. Rational Equations

Definition:

An equation that can be written in the form of $\frac{P(x)}{Q(x)} = 0$, where $P(x)$ and $Q(x)$ are polynomials, $Q(x) \neq 0.0$

Examples:

1. Identify rational equations

   a. $\frac{x^2+5lnx}{6x+7} = 0$ b. $3x^3 - 5x^2 + 6 = 2 - 5x^2$ c. $\frac{5x-6}{3^x} = 0$ d. $\frac{3-x^{-2}}{x^{-3}+1} = 0$

Solution:

a. $\frac{x^2+5lnx}{6x+7} = 0$ is not a rational equation because $x^2 + 5lnx$ is no polynomial.
b. $3x^3 - 5x^2 + 6 = 2 - 5x^2$ is a rational equation
c. $\frac{5x-6}{3^x} = 0$ is not a rational equation because 3^x is not polynomial.
d. $\frac{3-x^{-2}}{x^{-3}+1} = 0$ is a rational equation, because $\frac{3-x^{-2}}{x^{-3}+1} = 0$ can be written $\frac{x(3x^2-1)}{1+x^3} = 0$

Solving Rational Equations:

To solve rational equations of the form $\frac{P(x)}{Q(x)} = 0$, we just follow the steps

Step 1: restrict the domain, i. e $Q(x) \neq 0$

Step 2: solve the polynomial equation $P(x) = 0$

Step3: put the solution set

Example:

1. Find the solution set of the following equations

   a. $\frac{x^2-5x+6}{x^2-2} = 0$ b. $\frac{x}{x-1} = \frac{3}{x+1}$ c. $(\frac{5}{x-1} + \frac{1}{4-3x} = \frac{3}{6x-8}$ d. $\frac{x^2}{x+3} + \frac{3x}{x^2+x-6} = \frac{2x}{x+3}$

   e. $\frac{1+\frac{1}{x}}{x-\frac{1}{x}} = 0$ f. $\frac{x+4}{x-5} - \frac{1}{x+5} = \frac{10}{x^2-25}$
2. Two planes leave an airport flying at the same rate. If the first plane flies 1.4 hours longer than the second and travels 2700 km while the second plane travels only 2025km, for how long was each plane flying?
3. A tree casts a shadow of 24 m at a time when 2 m tall child casts a shadow of 1.5m. What is the height of the tree?
4. Two cars C_1 and C_2 start at the same time to make a trips from A to B and from B to A respectively. The two cars have constant speed of 70km/h and 80km/h respectively and the distance between A to B is 65km. if the two cars meet after t minutes, then find the value of t?

Solution:

a. $\frac{x^2-5x+6}{x^2-2}=0$

$\Rightarrow domain:\ x^2-2\neq 0$

$\Rightarrow x\neq \sqrt{2}\ or\ x\neq -\sqrt{2}$

$\Rightarrow\ x^2-5x+6=0$

$\Rightarrow (x-2)(x-3)=$

$\Rightarrow x=2\ or\ x=3$

Then $s.s=\{2\ ,3\ \}$

b. $\frac{x}{x-1}=\frac{2}{x+1}$

$\Rightarrow domain: x\neq 1,-1$

$\Rightarrow x(x+1)=2(x-1)$

$\Rightarrow x^2+x=2x-2$

$\Rightarrow x^2-x+2=0$

$b^2-4ac<0, it\ has\ no\ real\ solution.$

C. $\frac{5}{x-1}+\frac{1}{4-3x}=\frac{3}{6x-8}$

$domain: x\neq 1\ \ ,\frac{4}{3}$

$\Rightarrow \frac{5(4-3x)+(x-1)}{(x-1)(4-3x)}=\frac{-3}{2(4-3x)}$

$\Rightarrow\ \frac{5(4-3x)+(x-1)}{(x-1)}=\frac{-3}{2}$

$\Rightarrow \frac{20-15x+x-1}{x-1}=-\frac{3}{2}$

$\Rightarrow 2(19-14x)=3-3x$

$\Rightarrow 38-28x=3-3x$

$\Rightarrow 35=25x\Rightarrow x=\frac{7}{5}$

Then $s.s=\{\frac{7}{5}\}$

d. $\frac{x^2}{x+3}+\ \frac{3x}{x^2+x-\ 6}=\frac{2x}{x+3}$

$domain: x\neq -3{,}2$

$\Rightarrow \frac{x^2}{x+3}+\ \frac{3x}{(x-2)(x+3)}=\frac{2x}{x+3}$ LCM=$(x-2)(x+3)$

$\Rightarrow \frac{x^2(x-2)+3x}{(x-2)(x+3)}=\frac{2x(x-2)}{(x-2)(x+3)}$

$\Rightarrow x^2(x-2)+3x=2x(x-2)$

$\Rightarrow x^3-2x^2+3x=2x^2-4x$

$\Rightarrow x^3-2x^2-2x^2+3x+4x=0$

$\Rightarrow x^3-4x^2+7x=0$

$\Rightarrow x(x^2-4x+7)=0$

$\Rightarrow x=0\ or\ x^2-4x+7=0$

$\Rightarrow x=0\ or\ x^2-4x+7\neq 0$

$s.s=\{0\}$

e. $\frac{1+\frac{1}{x}}{x-\frac{1}{x}}=0$

Domain:$x\neq -1,0,1$

$\Rightarrow 1+\frac{1}{x}\div x-\frac{1}{x}=0$

$\Rightarrow \frac{x+1}{x}\ \ \div\frac{x^2-1}{x}=0$

$\Rightarrow \frac{x+1}{x}\times\frac{x}{x^2-1}=0$

$\Rightarrow \frac{x+1}{(x-1)(x+1)}=0$

$\Rightarrow \frac{1}{x-1}=0$

$\Rightarrow 1=0\ is\ false$

Therefore $s.s=\emptyset$

f. $\frac{x+4}{x-5}-\frac{1}{x+5}=\frac{10}{x^2-25}$

$\Rightarrow domain: x\neq -5{,}5$

$\Rightarrow \frac{(x+4)(x+5)-(x-5)}{(x-5)(x+5)}=\frac{10}{(x-5)(x+5)}$

$\Rightarrow (x+4)(x+5)-(x-5)=10$

$\Rightarrow x^2+9x+20-x+5-10=0$

$\Rightarrow x^2+8x+15=0$

$\Rightarrow (x+3)(x+5)=0$

$\Rightarrow x+3=0\ \ or\ x+5=0$

$\Rightarrow x=-3\ or\ x=-5$

$s.s=\{-5,-3\}$

2. Given the two planes are flying at the same rate
Let v_1 *be speed of the first plane* , v_2 *be speed of second plane.*
,s_1 , s_2 *be distance covered by plannes respectively* while t_1 *and* t_2 *be the time* of flight of the planes.

$$v_1 = \frac{s_1}{t_1} \quad and \quad v_2 = \frac{s_2}{t_2} \quad \Rightarrow v_1 = v_2 \ then \ \frac{s_1}{t_1} = \frac{s_2}{t_2}$$

From the question $s_1 = 2700km$, $s_2 = 2025km$ and $t_1 = 1.4\ hours + t_2$

,$\Rightarrow \frac{s_1}{t_1} = \frac{s_2}{t_2} \Rightarrow \frac{2700}{1.4+t_2} = \frac{2025}{t_2}$

,$domain: t \neq -1.4, 0$

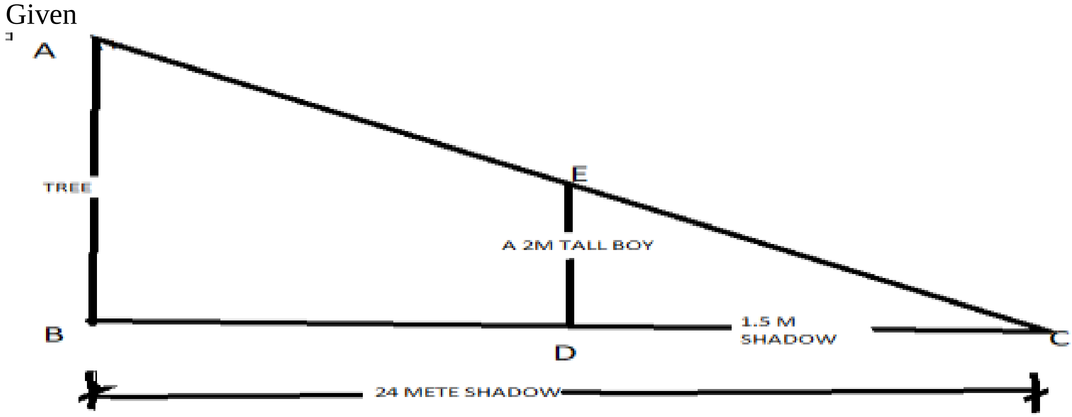
,$\Rightarrow 2700(t_2) = 2025(1.4 + t_2)$

, $\Rightarrow 2700t_2 - 2025t_2 = 2835 \Rightarrow 675t_2 = 2835 \Rightarrow t_2 = 4.2\ hours$

,$\Rightarrow t_1 = 1.4 + t_2 \Rightarrow t_1 = 1.4 + 4.2 = 5.6\ hours$

Answer: the first plane flies 5.6 hours and the second plane flies 4.2 hours to cover the given distances.

3. Given



From the figure, we can get similar triangles $\Delta ABC \sim \Delta EDC$ ……. AA similarity theorem.

Applying the property of similarity of triangles we have:

$\Rightarrow \frac{AB}{ED} = \frac{BC}{DC} = \frac{AC}{EC} \Rightarrow \frac{AB}{ED} = \frac{BC}{DC}$ ……. From side proportionality.

Let AB be x m long, then $\frac{AB}{ED} = \frac{BC}{DC}$

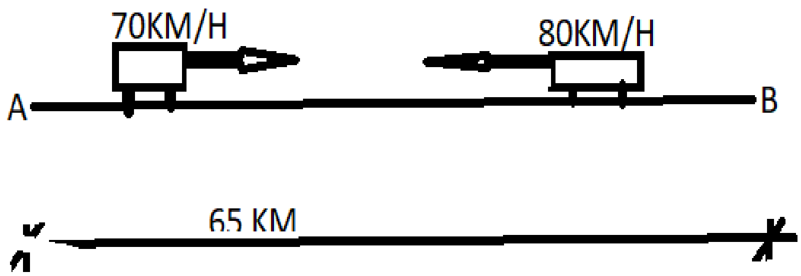
$\Rightarrow \frac{x}{2} = \frac{24}{1.5}$

Domain $x \in \mathbb{R}^+$

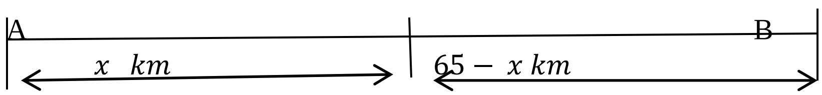
,$1.5x = 2(24) \Rightarrow x = 32$

Answer: the height of the tree is 32m.

4. Given C_1 and C_2 be two cars



, The two cars are starting their trip at the same time and $distance\ s = 65km$
Let two cars start at $t_0 = 0hours, they\ meet\ at\ \ a\ distance\ of\ xkm\ from\ first\ car$ and $65 - x\ km$ from the second car at t hours.

i.e. 

But $t = \frac{x}{70} = \frac{65-x}{80} \Rightarrow 80x = 70(65 - x)$ where $0 \leq x \leq 65$
, $\Rightarrow 80x = 4550 - 70x \Rightarrow 80x + 70x = 4550$
,$\Rightarrow 150x = 4550 \Rightarrow x = \frac{4550}{150} = 30.33$
,$\Rightarrow x = 30.33km$ and $t = \frac{x}{70} = \frac{30.33}{70} = 0.43\ hours$
Therefore the two cars meet at t = 0.43 hors

1.3.2. Rational Inequalities

Definition: A rational inequality is an inequality which can be written of the form:

$\frac{p(x)}{q(x)} \leq 0\ \ or\ \ \frac{p(x)}{q(x)} \geq 0$ Where $p(x) and\ q(x) are\ polynomial\ expressions\ , q(x) \neq 0$

Example:

Which of the following are rational inequalities?

a. $\frac{35x+6}{6-25x} < 25 - 14x$ b. $3^x > \frac{58-x}{5x}$ c. $\frac{5x+sinx}{8x+6} \geq 0$ d. $\frac{6-21x^2+1}{x^2+5x+6} \geq 0$

Solution:

a) $\frac{35x+6}{6-25x} < 25 - 14x$, $\Rightarrow 35x + 6, 6 - 25x$ and $25 - 14x$ are polynomials then it is a rational inequality.

b) $3^x > \frac{58-x}{5x}$ is not a rational inequality, since $3^x is\ a\ polynomial.$

c) . $\frac{5x+sinx}{8x+6} \geq 0\ is\ not\ a\ rational\ inequality$, since $\sin x\ is\ not\ polynomial$

d) $\frac{6-21x^2+1}{x^2+5x+6} \geq 0$ *is a rational inequality*

Solving Rational Inequalities

Note:

let $\frac{\boldsymbol{P(x)}}{\boldsymbol{Q(x)}}$ be a rational expression then

Case 1. If $\frac{P(x)}{Q(x)} > 0, then\ P(x) and\ Q(x) have\ the\ same\ sign$

i.e. $i.\ P(x) > 0\ and\ Q(x) > 0\ or$

ii $.P(x) < 0\ and\ Q(x) < 0$

Case2. If $\frac{P(x)}{Q(x)} < 0, then\ P(x) and\ Q(x) have\ opposite\ sign$

i.e. $i. P(x) > 0\ and\ Q(x) < 0\ or$

ii. $P(x) < 0\ and\ Q(x) > 0$

Examples:

1. Solve $\frac{x}{x-1} > 0$

2. *solve* $\frac{x+2}{x-1} < 0$

Solution

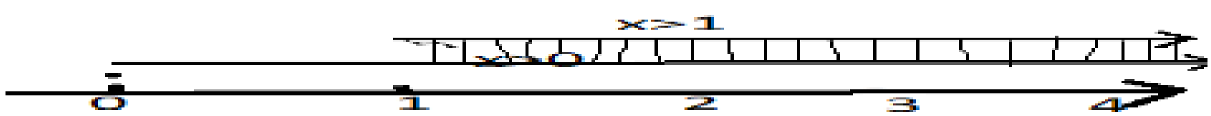
1. Use case 1

Domain $= R/\{1\}$

$, \Rightarrow \frac{x}{x-1} > 0$

$, \Rightarrow i. x > 0\ and\ x - 1 > 0$

Take the intersection of the two inequalities

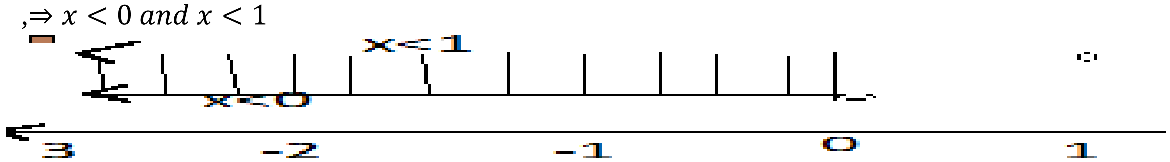


The intersection:$x > 0\ \cap x > 1 = x > 1$

Therefore $ss_1 = \{x: x > 1\} = (1,\ \infty)$

, ii. $x < 0\ and x - 1 < 0$

$,\Rightarrow x < 0\ and\ x < 1$



intersection: $x < 0 \cap x < 1 = x < 0$

$,ss_2 = \{x\, x < 0\} = (-\infty\,,0)$

The solution is the union of two sets

$,\Rightarrow ss = ss_1 \cup ss_2 = \{x: x < 0\} \cup \{x: x > 1\} = \{x: x < 0 \; or \; x > 1$

2. $\frac{x+2}{x-1} < 0$

   i. $x + 2 > 0 \; and \; x - 1 < 0 \Rightarrow x > -2 \; and \; x < 1$

   $\Rightarrow x > -2 \;\cap x < 1 = -2 < x < 1$

   $\Rightarrow ss_1 = -2 < x < 1 = (-2\,,1\,)$

   ii. $x + 2 < 0 \; and \; x - 1 > 0 \Rightarrow x < -2 \; and \; x > 1$

   $\Rightarrow x < -2 \;\cap x > 1 = \{\,\}$

   $\Rightarrow ss_2 = \emptyset$

   Therefore the solution set becomes

   $\Rightarrow ss = ss_1 \cup ss_2 == (-2\,,1\,) \cup \{\,\} = (-2\,,1)$

General steps solving rational inequalities using sign chart

Step 1: restrict the domain

Step 2: factorize the numerators and denominators

Step3: determine the boundary points on a number line to get intervals on the sign chart

Step 4: check the sign of each factor on each boundary

Step5: put the solution according to the sign of inequality by observing the sign if last row

Example :

1. Solve the inequalities using sign chart

   a. $\frac{x+2}{x-1} \geq x + 2$ b. $\frac{2x^2+x-1}{x^2-4x+4} \geq 0$ c. $\frac{1}{x} \leq \frac{x}{x-1} + 1$ d. $\frac{x^2}{x^2+2} \geq 0$

   e. $\frac{x^2+4}{x^2} < 0$ f. $\frac{1}{3} + \frac{2}{x^2} \leq \frac{5}{3x}$ g. $\frac{27-2x}{3} + x \leq \frac{x-4}{2} - 9$ h. $\frac{1-5x}{x+1} \leq \frac{9-3x}{x-3}$

Solution:

a. $\frac{x+2}{x-1} \geq x + 2$

Step 1: Domain $= R/\{1\}$

Re- arrange the inequality

$\Rightarrow \frac{x+2}{x-1} - (x + 2) \geq 0 \Rightarrow, \frac{x+2-(x+2)(x-1)}{x-1} \geq 0$ ……LCM $= x - 1$

$\Rightarrow \frac{x+2-(x^2-x+2x-2)}{x-1} \geq 0 \quad \Leftrightarrow \frac{x+2-x^2-x+2}{x-1} \geq 0$

Step2: $\Leftrightarrow \frac{x^2-4}{x-1} \leq 0 \quad \Leftrightarrow \frac{(x-2)(x+2)}{x-1} = 0$

Step3: The boundary points are at $x = -2, 1$ and 2

Step4: construct sign chart

	$x < -2$	$x = -2$	$-2 < x < 1$	$x = 1$	$1 < x < 2$	$x = 2$	$x > 2$
$x^2 - 4$	+	0	-	-	-	0	+
$x - 1$	-	-	-	0	+	-	+
$\frac{x^2 - 4}{x - 1}$	-	0	+	∄	-	0	+

The symbol ∄ in the table shows the set does not include that point or open interval

- ❖ Solution set contains the intervals with negative sign on the last row.

.$ss = (-\infty, -2] \cup (1, 2]$

b. $\frac{2x^2+x-1}{x^2-4x+4} \geq 0 \quad \Leftrightarrow \frac{(2x-1)(x+1)}{(x-2)(x-2)} \geq 0$

domain $= R/\{2\}$

Boundary points $x = -1, \frac{1}{2}$ and

Sign chart

	$x < -1$	$x = -1$	$-1 < x < 1/2$	$x = \frac{1}{2}$	$\frac{1}{2} < x < 2$	$x = 2$	$x > 2$
$2x^2 + x - 1$	+	0	-	0	+	+	+
$x^2 - 4x + 4$	+	+	+	+	+	0	+
$\frac{2x^2 + x - 1}{x^2 - 4x + 4}$	+	0	-	0	+	∄	+

$$ss = (-\infty, -1] \cup \left[\frac{1}{2}, 2\right) \cup (2. \infty)$$

c. $\frac{1}{x} \leq \frac{x}{x-1} + 1 \Leftrightarrow . \frac{1}{x} - \frac{x}{x-1} - 1 \leq 0$

domain $= R/\{0, 1\}$

$\Leftrightarrow \frac{1}{x} - \frac{x}{x-1} - 1 \leq 0 \Leftrightarrow (\frac{x-1-x^2-(x(x-1))}{x(x-1)} \leq 0$

$\Leftrightarrow \frac{x - 1 - x^2 - x^2 + x}{x(x-1)} \leq 0$

$\Leftrightarrow \frac{-2x^2+2x-1}{x(x-1)} \leq 0$

$-2x^2 + 2x - 1$ $doesn't\ be\ factorized$

The boundary points are at $x = 0, 1$

Sign chart

	$x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$x > 1$
$-2x^2 + 2x - 1$	-	-	-	-	-
$x(x-1)$	+	0	-	0	+
$\frac{-2x^2+2x-1}{x(x-1)}$	-	∄	+	∄	-

$$ss = (-\infty, 0) \cup (1, \infty)$$

$d.\ \frac{x^2}{x^2+2} \geq 0$

$\Leftrightarrow \frac{x(x)}{x^2+2} \geq 0$

domain $= R$

Boundary points at $x = 0$ since $x^2 + 2 > 0$

Sign chart

	$x < 0$	$x = 0$	$x > 0$
x^2	+	**0**	+
$x^2 + 2$	+	+	+
$\frac{x^2}{x^2+2}$	+	**0**	+

The solution set $ss = \mathbb{R}$ since the last row is either zero or positive

$e.\ \frac{x^2+4}{x^2} < 0$ domain $= R/\{0\}$

The boundary point is at$x = 0$ but $x^2 + 4$ cannot be factorized

Sign chart

	$x < 0$	$x = 0$	$x > 0$
$x^2 + 2$	+	+	+
x^2	+	0	+
$\frac{x^2+4}{x^2}$	+	∄	+

The solution set= ∅

e) $\frac{1}{3} + \frac{2}{x^2} \leq \frac{5}{3x}$ $domain = R/\{0\}$

,$\frac{1}{3} + \frac{2}{x^2} - \frac{5}{3x} \leq 0 \Leftrightarrow \frac{x^2+3(2)-5(x)}{3x^2} \leq 0 \Leftrightarrow \frac{x^2-5(x)+6}{3x^2} \leq 0$

.$\frac{(x-2)(x-3)}{3x^2} \leq 0$

Boundary points at $x = 0, 2, 3$

Sign chart

	$x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$2 < x < 3$	$x = 3$	$x > 3$
$x^2 - 5(x) + 6$	+	+	+	0	-	0	+
$3x^2$	+	0	+	+	+	+	+
$\frac{x^2 - 5(x) + 6}{3x^2}$	+	$\nexists$	+	0	-	0	+

The solution set $ss = [2\,,3]$

f) $\frac{27-2x}{3} + x \leq \frac{x-4}{2} - 9 \Leftrightarrow \frac{27-2x}{3} + x - \left(\frac{x-4}{2} - 9\right) \leq 0$ domain $= R$

,$\Leftrightarrow \frac{27-2x+3x}{3} - \frac{x-4-18}{2} \leq 0 \Leftrightarrow \frac{27+x}{3} - \frac{x-22}{2} \leq 0$

Now $\frac{2(27+x)-\left(3(x-22)\right)}{6} \leq 0 \Leftrightarrow \frac{(120-x)}{6} \leq 0$

Boundary at $x = 120$

	$x < 120$	$x = 120$	$x > 120$
$(120 - x)$	+	0	-
6	+	+	+
$\frac{(120 - x)}{6}$	+	0	-

Solution $ss = [120, \infty)$

g) Left as an exercise for you!!!! solution$ss = (-\infty\,, -1) \cup [2\,, \infty)$