Lesson 1: Concept of Matrices

Definition:
Let m and n be positive integers. A rectangular array of numbers in $\mathbb{R}$ of the form:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$ is an m by n or $m\ x\ n$ matrix.

Where m: shows the horizontal rows and
n: shows the vertical columns

Note: 1.

i. the i[th] row of A = $(a_{i1} a_{i2} a_{i3} a_{i4} \ldots a_{in})$

ii. The j[th] column of A = $\begin{pmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{pmatrix}$

iii. The real number a_{il} *is the* $(i, j) - $ *entry or element of* A.

iv. The Matrix A is written as $A = (a_{ij})_{mn}$

v. $m \times n$ *is the size or order of matrix* A.

Examples:

1. Let matrix A = $\begin{pmatrix} 1 & 5 & 6 & 3 \\ 2 & 8 & -1 & 0 \\ 4 & -9 & 12 & 10 \end{pmatrix}$ then find
   a. number of rows and columns
   b. a_{23}
   c. a_{33}
   d. Size or order of the matrix
   e. entry(3,4)
2. Find: size and Elements for the matrix
   a. B= $(5 \quad 9 \quad -3.5 \quad 0.897 \quad)$,
   b. $A = \begin{pmatrix} 4 \\ 8 \\ 0 \end{pmatrix}$
   c. $C = (5)$

Solution:

a. Matrix A has 3- rows and 4- columns

b. a_{23} : 2- stands for the 2^{nd} row and 3- stands for the 3^{rd} column, the intersection number of 2nd row and 3rd column is a_{23} = -1

c. $a_{23} = intersection\ number\ of$ 3rd row and 3rd column is 12.

d. Size $of\ A = 3by\ 4\ or\ 3 \times 4$

e. .entry(3,4)= $a_{34} = 10$

2. a. Matrix B has 1- row and 3- columns,

   - Size of B $= 1 \times 3$
   - Elements of B: - $b_{11} = 5,$ $b_{12} = 9$,$b_{13} = -3.5$ $b_{14} = 0.897$

   b. matrix A has 3- rows and 1- column:

   - Size of A $= 3 \times 1$
   - Elements of A ; $a_{11} = 4$, $a_{21} = 8\ and\ a_{31} = 0$

   c.matrix C has 1-row and 1- column

   - Size of C= 1×1
   - Elements of C : $a_{11} = 5$, (matrix C has only one entry)

3. Three students Selam, Hagos and Chaltu have a number of 10, 50 and 25 cents coin in their pocket. The following table shows what they have:

	Students name			
No of coins		Selam	Hagos	Chaltu
	10 cents coin	2	6	4
	50 cents coin	3	2	0
	25 cents coin	4	0	5

Find

a. write the matrix form of the problem
b. size of the matrix
c. what does $a_{31} and\ a_{23}$ tells

Solution:

a. let A denoting number of coins they have

$$A = \begin{pmatrix} 2 & 6 & 4 \\ 3 & 2 & 0 \\ 4 & 0 & 5 \end{pmatrix}$$

b. size of A= 3×3

c. $a_{31} = 4,\ tells\ Selam\ has\ four 25\ cent\ coins$
, $a_{23} = 0$, $tells\ Chaltu\ has\ no\ 50 cent\ coin.$

4. Let A= $(a_{ij})_{23} and\ a_{ij} = j - i$, then find matrix A?

5. Construct a 3×4 matrix A= $(a_{ij})_{34}, where\ a_{ij} = 3i - 2j$

Solution:

4.let A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

$a_{ij} = j - i, \implies a_{11} = 1 - 1 = 0,$ $\quad a_{21} = 1 - 2 = -1$

. $a_{12} = 2 - 1 = 1$ $\quad a_{22} = 2 - 2 = 0$

, $a_{31} = 3 - 1 = 2$ $\quad a_{23} = 3 - 2 - 1$

Therefore A = $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$

5.Let A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix}$ and $a_{ij} = 3i - 2j$

Then $a_{ij} = 3i - 2$

,$\implies a_{11} = 3 * 1 - 2 * 1 = 3 - 2 = 1$ $\quad a_{31} = 3 * 3 - 2 * 1 = 9 - 2 = 7$

, $a_{12} = 3 * 1 - 2 * 2 = 3 - 4 = -1$ $\quad a_{32} = 3 * 3 - 2 * 2 = 9 - 4 = 5$

, $a_{13} = 3 * 1 - 2 * 3 = 3 - 6 = -3$ $\quad a_{33} = 3 * 3 - 2 * 3 = 9 - 6 = 3$

, $a_{14} = 3 * 1 - 2 * 4 = 3 - 8 = -5$ $\quad a_{34} = 3 * 3 - 2 * 4 = 9 - 8 = 1$

$a_{21} = 3 * 2 - 2 * 1 = 6 - 2 = 4$

$a_{22} = 3 * 2 - 2 * 2 = 6 - 4 = 2$

$a_{23} = 3 * 2 - 2 * 3 = 6 - 6 = 0$

$a_{24} = 3 * 2 - 2 * 4 = 6 - 8 = -2$

Then matrix A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{pmatrix} = \begin{pmatrix} 1 & -1 & -3 & -5 \\ 4 & 2 & 0 & 2 \\ 7 & 5 & 3 & 1 \end{pmatrix}$

1.1. Types Of Matrices

There are different types of matrices:

1. Column matrix(column vector):
   - A matrix having only one column
   - A= $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$ is a column matrix
2. Row matrix (row vector):
   - A matrix having only one row
   - A= $(a_1 \quad a_2 \quad \cdots \quad a_n)$ is a row matrix
3. Zero matrix:
   - A matrix with all zero entries
   - A $= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
4. Square matrix:
   - A matrix in which the number of rows is equal to number of columns.

- A= $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ is a 2×2 square matrix
- A= $\begin{pmatrix} 2 & 6 & 4 \\ 3 & 2 & 0 \\ 4 & 0 & 5 \end{pmatrix}$ is a 3×3 $square\ matrix$

5. Diagonal matrix:
   - A square matrix with all zero entries except diagonal entries.
   - A= $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{pmatrix}$ is a diagonal matrix
6. Scalar matrix :
   - A diagonal matrix all diagonal entries are equal.
   - A= $\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$ is a scalar matrix
7. Identity matrix:
   - A scalar or diagonal matrix with all the diagonal entries are equal to 1.
   - A= $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is an identity matrix
8. Triangular matrix:
   a. Upper Triangular Matrix:
      - A square matrix whose entries below the main diagonal are all zero.
      - A= $\begin{pmatrix} 2 & 6 & 4 \\ 0 & 2 & 1 \\ 0 & 0 & 5 \end{pmatrix}$ is an upper triangular matrix

   b. Lower Triangular Matrix :
      - A square matrix whose entries above the main diagonal are all zero.
      - A= $\begin{pmatrix} 2 & 0 & 0 \\ 3 & 2 & 0 \\ 4 & 0 & 5 \end{pmatrix}$ is a lower triangular matrix

Equality of Matrices

Definition:

$A = (a_{ij})_{mn}$ and $B = (b_{ij})_{pq}$ are said to be equal if and only if

- They have the same size i.e. $m = p$ and $n = q$
- The corresponding entries are equal i.e. $a_{ij} = b_{ij}$

Example:

1. Identify the following matrices are equal or not?
   a. A= $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ and $B = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$

   b. A= $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$

2. Find the values of x, y, a and b if

a. $\begin{pmatrix} 3x+4y & 6 & x-2y \\ a+b & 2a-b & -3 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 4 \\ 5 & -5 & 3 \end{pmatrix}$

b. $\begin{pmatrix} x^2-1 & 1 & 2 \\ -1 & a+b^2 & -1 \end{pmatrix} = \begin{pmatrix} 2x-2 & 1 & 2 \\ -1 & -3a+b^2+12 & -1 \end{pmatrix}$

Solution:

1. a. A= $\begin{pmatrix} 2 & 3 \\ 4 & 5 \end{pmatrix}$ $and\ B = \begin{pmatrix} 5 & 3 \\ 4 & 2 \end{pmatrix}$

Matrices A$\neq B\ since\ 2 \neq 5\ and\ 5 \neq 2$

b. A= $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$ $and\ B = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -1 \end{pmatrix}$

Matrices $A = B$

a. $\begin{pmatrix} 3x+4y & 6 & x-2y \\ a+b & 2a-b & -3 \end{pmatrix} = \begin{pmatrix} 2 & 6 & 4 \\ 5 & -5 & 3 \end{pmatrix}$ if and only if

The following are satisfied:

, $3x+4y=2\ and\ x-2y=4$ similarly $a+b=5\ and\ 2a-b=-5$

,$\begin{cases} 3x+4y=2 \\ x-2y=4 \end{cases}$ $\begin{cases} a+b=5 \\ 2a-b=-5 \end{cases}$

,$2*\begin{cases} 3x+4y=2 \\ 2x-4y=8 \end{cases}$ $3a=0 \Longrightarrow a=0$

,$5x=10 \Longrightarrow x=2$ $b=5-a=5$

,$y=\frac{x-4}{2}=\frac{2-4}{2}=-2$ $\therefore a=0\,, b=5$

$\therefore x=2, y=-2$

$b.$ $\begin{pmatrix} x^2-1 & 1 & 2 \\ -1 & a+b^2 & -1 \end{pmatrix} = \begin{pmatrix} 2x-2 & 1 & 2 \\ -1 & -3a+b^2+12 & -1 \end{pmatrix}$

,$x^2-1=2x-2$ and $a+b^2=-3a+b^2+12$

,$x^2-1-2x+2=0$ $a+b^2+3a-b^2=12$

,$x^2-2x+1=0$ $4a=12$

,$(x-1)^2=0$ $\Longrightarrow a=3\ for\ any\ b \in \mathbb{R}$

,$\Longrightarrow x=1$ $a=3\ and\ b \in \mathbb{R}$

Lesson 2: Operations on Matrices

i. Addition on matrices
ii. Subtraction on matrices
iii. Multiplication

i. Addition Of Matrices

Let $A = (a_{ij})_{mn}\ and\ B = (b_{ij})_{mn} be\ two\ matrices\ of\ the\ same\ order$, then their sum or difference denoted by $A \pm B = (a_{ij})_{mn} \pm (b_{ij})_{mn} = (a_{ij} \pm b_{ij})_{mn}$ where $m, n \in Z^+, a, b \in R$

Example:

1. Let $A = \begin{pmatrix} 3 & 5 & 2.1 \\ 4 & 2 & 6 \\ 0 & -1 & 5 \end{pmatrix}$ $and\ B = \begin{pmatrix} 7 & 6 & 5 \\ 10 & 2 & 2.15 \\ 9 & 8 & -6.5 \end{pmatrix}$ be matrices , then find their sum

2. For the given matrices A and B below determine their sum

$$A = \begin{pmatrix} a+b & 3c & b-2a \\ d-c & 5d & 41 \\ 2a & 9 & a+5 \end{pmatrix} \text{ and } B = \begin{pmatrix} b & 3c & 2a \\ d-c & -5d & 1 \\ -2a & 9 & 5-b \end{pmatrix}$$

Solution:

1. A + B $= \begin{pmatrix} 3 & 5 & 2.1 \\ 4 & 2 & 6 \\ 0 & -1 & 5 \end{pmatrix} + \begin{pmatrix} 7 & 6 & 5 \\ 10 & 2 & 2.15 \\ 9 & 8 & -6.5 \end{pmatrix} = \begin{pmatrix} 3+7 & 5+6 & 2.1+5 \\ 4+10 & 2+2 & 6+2.15 \\ 0+9 & -1+8 & 5-6.5 \end{pmatrix}$

$$= \begin{pmatrix} 10 & 11 & 7.1 \\ 14 & 4 & 8.15 \\ 9 & 7 & -1.5 \end{pmatrix}$$

2. A + B $= \begin{pmatrix} a+b & 3c & b-2a \\ d-c & 5d & 41 \\ 2a & 9 & a+5 \end{pmatrix} + \begin{pmatrix} b & 3c & 2a \\ d-c & -5d & 1 \\ -2a & 9 & 5-b \end{pmatrix}$

$$= \begin{pmatrix} a+b+b & 3c+3c & b-2a+2a \\ d-c+d-c & 5d+(-5d) & 41+1 \\ 2a+(-2a) & 9+9 & a+5+5-b \end{pmatrix}$$

$$= \begin{pmatrix} a+2b & 6c & b \\ 2d-2c & 0 & 42 \\ 0 & 18 & a-b+10 \end{pmatrix}$$

ii. Subtraction of Matrices

> Let $A = (a_{ij})_{mn}$ $and\ B = (b_{ij})_{mn}$ $be\ two\ matrices\ of\ the\ same\ order$, then:
> their difference denoted by $A - B = (a_{ij})_{mn} - (b_{ij})_{mn} = (a_{ij} - b_{ij})_{mn}$ where $m, n \in Z^{+}, a, b \in R$

Example:

2.Let $A = \begin{pmatrix} 3 & 0 & 8 \\ 4 & 4 & 6 \\ 0 & -1 & 5 \end{pmatrix}$ $and\ B = \begin{pmatrix} 7 & 6 & 5 \\ 10 & -8 & 9 \\ 9 & 12 & -11 \end{pmatrix}$ Then find their difference

Solution:

2. A - B $= \begin{pmatrix} 3 & 0 & 8 \\ 4 & 4 & 6 \\ 0 & -1 & 5 \end{pmatrix} - \begin{pmatrix} 7 & 6 & 5 \\ 10 & -8 & 9 \\ 9 & 12 & -11 \end{pmatrix} = \begin{pmatrix} 3-7 & 0-6 & 8-5 \\ 4-10 & 4-(-8) & 6-9 \\ 0-9 & -1-12 & 5-(-11) \end{pmatrix}$

$$= \begin{pmatrix} -4 & -6 & 3 \\ -6 & 12 & -3 \\ -9 & -13 & 16 \end{pmatrix}$$

Properties of Addition of Matrices

Let A , B and C be matrices of real numbers of the same order, then the following properties are satisfied

- A + B = B + A ………………………… Commutative property
- A + (B + C) = (A +B) + C…………….Associative property
- A + 0 = A = 0 + A ……………………..Existence of additive identity
- A + (- A) = 0 ………………………… Existence of additive inverse

Examples:

1. Let A = $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix}$, B = $\begin{pmatrix} -3 & 6 \\ 1 & -2 \end{pmatrix}$, C = $\begin{pmatrix} -3 & 7 \\ 5 & 2 \end{pmatrix}$ and D = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ find
   a. A +B
   b. B + A
   c. (A + B) + C
   d. A + (B + C)
   e. A +D
   f. A + (-A)

Solution:

a. A + B = $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} -3 & 6 \\ 1 & -2 \end{pmatrix} = \begin{pmatrix} -1 & 7 \\ 5 & 1 \end{pmatrix}$

b. B + A = $\begin{pmatrix} -3 & 6 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} -1 & 7 \\ 5 & 1 \end{pmatrix}$

∴ A +B = B + A since addition is commutative

c. (A + B) + C = $\left(\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} -3 & 6 \\ 1 & -2 \end{pmatrix}\right) + \begin{pmatrix} -3 & 7 \\ 5 & 2 \end{pmatrix}$

$$= \begin{pmatrix} 2 + (-3) & 1 + 6 \\ 4 + 1 & 3 + (-2) \end{pmatrix} + \begin{pmatrix} -3 & 7 \\ 5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 7 \\ 5 & 1 \end{pmatrix} + \begin{pmatrix} -3 & 7 \\ 5 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -4 & 14 \\ 10 & 3 \end{pmatrix}$$

d. A+ (B + C) = $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} + \left(\begin{pmatrix} -3 & 6 \\ 1 & -2 \end{pmatrix} + \begin{pmatrix} -3 & 7 \\ 5 & 2 \end{pmatrix}\right)$

$$= \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} -3 + (-3) & 6 + 7 \\ 1 + 5 & -2 + 2 \end{pmatrix}$$

$= \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} -6 & 13 \\ 6 & 0 \end{pmatrix}$

$= \begin{pmatrix} 2+(-6) & 1+13 \\ 4+6 & 3+0 \end{pmatrix}$

$= \begin{pmatrix} -4 & 14 \\ 10 & 3 \end{pmatrix}$

∴ (A + B) + C = A+ (B + C) since addition is associative

e. A +D $= \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 2+0 & 1+0 \\ 4+0 & 3+0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = A$

∴ D = $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ an additive identity of 2by2 matrices

f. A+ (-A) = A – A = $\begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 1 \\ 4 & 3 \end{pmatrix} = \begin{pmatrix} 2-2 & 1-1 \\ 4-4 & 3-3 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

iii. Multiplication of matrices

a. Scalar Multiplication

Definition:
Let $A = (a_{ij})_{mn}$ *be any matrix and* $r \in R$, *then* $rA = r(a_{ij})_{mn} = (ra_{ij})_{mn}$

Example:

1. Let A = $\begin{pmatrix} 3 & 0 & 8 \\ 4 & 4 & 6 \\ 0 & -1 & 5 \end{pmatrix}$ be a 3 by 3 matrix then find:
   a. 6A b. $\frac{1}{2}A$ c. -2(3A)
2. Given A= $\begin{pmatrix} 1 & 0 & -2 \\ 1 & 2 & 3 \end{pmatrix}$ *and* $B = \begin{pmatrix} -4 & 2 & 0 \\ -1 & 1 & 3 \end{pmatrix}$, then find
   a. 2A + 3B b. (1+3)A
3. Given $A = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 3 & -1 & 1 \end{pmatrix}$ *and* $B = \begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix}$, find matrix C that satisfy A + 2C =3B

Solution:

a. 6A = $6\begin{pmatrix} 3 & 0 & 8 \\ 4 & 4 & 6 \\ 0 & -1 & 5 \end{pmatrix} = \begin{pmatrix} 6\times3 & 6\times0 & 6\times8 \\ 6\times4 & 6\times4 & 6\times6 \\ 6\times0 & 6\times-1 & 6\times5 \end{pmatrix} = \begin{pmatrix} 18 & 0 & 48 \\ 24 & 24 & 36 \\ 0 & -6 & 30 \end{pmatrix}$

b. $\frac{1}{2}A = \frac{1}{2}\begin{pmatrix} 3 & 0 & 8 \\ 4 & 4 & 6 \\ 0 & -1 & 5 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\times3 & \frac{1}{2}\times0 & \frac{1}{2}\times8 \\ \frac{1}{2}\times4 & \frac{1}{2}\times4 & \frac{1}{2}\times6 \\ \frac{1}{2}\times0 & \frac{1}{2}\times-1 & \frac{1}{2}\times5 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 0 & 4 \\ 2 & 2 & 3 \\ 0 & -\frac{1}{2} & \frac{5}{2} \end{pmatrix}$

c. $-2(3A) = -2\left(3\begin{pmatrix}3 & 0 & 8\\4 & 4 & 6\\0 & -1 & 5\end{pmatrix}\right) = -2\begin{pmatrix}3\times 3 & 3\times 0 & 3\times 8\\3\times 4 & 3\times 4 & 3\times 6\\3\times 0 & 3\times -1 & 3\times 5\end{pmatrix}$

$= -2\begin{pmatrix}9 & 0 & 24\\12 & 12 & 18\\0 & -3 & 15\end{pmatrix} = \begin{pmatrix}-2\times 9 & -2\times 0 & -2\times 24\\-2\times 12 & -2\times 12 & -2\times 18\\-2\times 0 & -2\times -3 & -2\times 15\end{pmatrix}$

$= \begin{pmatrix}-18 & 0 & -48\\-24 & -24 & -36\\0 & 6 & -30\end{pmatrix}$

1. a. $2A + 3B = 2\begin{pmatrix}1 & 0 & -2\\1 & 2 & 3\end{pmatrix} + 3\begin{pmatrix}-4 & 2 & 0\\-1 & 1 & 3\end{pmatrix} = \begin{pmatrix}2 & 0 & -4\\2 & 4 & 6\end{pmatrix} + \begin{pmatrix}-12 & 6 & 0\\-3 & 3 & 9\end{pmatrix}$

$= \begin{pmatrix}-10 & 6 & -4\\-1 & 7 & 15\end{pmatrix}$

b. $(1+3)A = 4A = 4\begin{pmatrix}1 & 0 & -2\\1 & 2 & 3\end{pmatrix} = \begin{pmatrix}4 & 0 & -8\\4 & 8 & 12\end{pmatrix}$

2. $A + 2C = 3B \Leftrightarrow 2C = 3B - A$

$,\Leftrightarrow C = \frac{1}{2}(3B - A) = \frac{1}{2}\left(3\begin{pmatrix}3 & -1 & 2\\4 & 2 & 5\\2 & 0 & 3\end{pmatrix} - \begin{pmatrix}1 & 2 & -3\\5 & 0 & 2\\3 & -1 & 1\end{pmatrix}\right)$

$= \frac{1}{2}\left(\begin{pmatrix}9 & -3 & 6\\12 & 6 & 15\\6 & 0 & 9\end{pmatrix} - \begin{pmatrix}1 & 2 & -3\\5 & 0 & 2\\3 & -1 & 1\end{pmatrix}\right)$

$= \frac{1}{2}\left(\begin{pmatrix}9-1 & -3-2 & 6-(-3)\\12-5 & 6-0 & 15-2\\6-3 & 0-(-1) & 9-1\end{pmatrix}\right)$

$= ½\begin{pmatrix}8 & -5 & 9\\7 & 6 & 13\\3 & 1 & 8\end{pmatrix}$

$= \begin{pmatrix}4 & -\frac{5}{2} & \frac{9}{2}\\\frac{7}{2} & 0 & \frac{13}{2}\\\frac{3}{2} & \frac{1}{2} & 4\end{pmatrix}$

Properties of Scalar Multiplication of Matrices

Let A and B be matrices of real numbers of the same order, $r, s \in R$ then the following properties hold:

- $r(A + B) = rA + r$**B………………. Distributive property**
- $(r + s)A = rA + sA$**……………… Distributive property**
- $(rs)A = r(sA)$ **…………………… Associative property**

c. Multiplication of Matrices

Let $A = (a_{ij})_{mn}$ $and\ B = (b_{ij})_{pq}$ be matrices, then the product AB exists
If and only if:

- The number of column of A is equal to number of rows of B.
- i.e. n = p
- order of AB is $m \times q$

Examples:

1. identify which pair of matrices are comfortable for multiplication
   a. A $= \begin{pmatrix} 4 & 0 & -8 \\ 4 & 8 & 12 \end{pmatrix}$ $and\ B = \begin{pmatrix} 42 & 0 & 2 \\ 0 & 8 & 5 \end{pmatrix}$
   b. A = $\begin{pmatrix} 2 & 0 & -8 \\ 3 & 5 & 1 \end{pmatrix}$ and B = $\begin{pmatrix} 2 & 3 \\ 0 & 5 \\ 8 & 1 \end{pmatrix}$
   c. A = $\begin{pmatrix} a & b & c & d \end{pmatrix}$ and B = $\begin{pmatrix} e & i & m \\ f & j & n \\ g & k & p \\ h & l & q \end{pmatrix}$,
   Where $a, b, c, d, e, f, g, h, I, j, k, l, m, n, p, q \in \mathbb{R}$

Solution:

1. a. A $= \begin{pmatrix} 4 & 0 & -8 \\ 4 & 8 & 12 \end{pmatrix}_{23}$ $B = \begin{pmatrix} 42 & 0 & 2 \\ 0 & 8 & 5 \end{pmatrix}_{23}$
   A and B are not multiplicative comfortable i.e. $AB\ doesn't\ exist,$ since the number of columns of A is different from number of rows of B.
   b. A = $\begin{pmatrix} 2 & 0 & 8 \\ 3 & 5 & 1 \end{pmatrix}_{23}$ and B = $\begin{pmatrix} 2 & 3 \\ 0 & 5 \\ 8 & 1 \end{pmatrix}_{32}$
   A and B are comfortable .i.e. AB exists, since the number of columns of A is equal to number of rows of B.

   $c.$ A = $\begin{pmatrix} a & b & c & d \end{pmatrix}$ and B = $\begin{pmatrix} e & i & m \\ f & j & n \\ g & k & p \\ h & l & q \end{pmatrix}$,

   Where $a, b, c, d, e, f, g, h, I, j, k, l, m, n, p, q \in \mathbb{R}$
   - A and B are comfortable i.e. AB exists, since the number of columns of A is equal to the number of rows of B

Multiplication Rules:

Let A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}_{2\times2}$ $and\ B = \begin{pmatrix} w & x & r \\ h & y & s \end{pmatrix}_{2\times3}$ $be\ matrices$ then

$$AB = \begin{pmatrix} R_1C^1 & R_1C^2 & R_1C^3 \\ R_2C^1 & R_2C^2 & R_2C^3 \end{pmatrix}_{2\times3} = \begin{pmatrix} (a\ b)\binom{w}{h} & (a\ b)\binom{x}{y} & (a\ b)\binom{r}{s} \\ (c\ d)\binom{w}{h} & (c\ d)\binom{x}{y} & (c\ d)\binom{r}{s} \end{pmatrix}_{2\times3}$$

$$= \begin{pmatrix} aw + bh & ax + by & ar + bs \\ cw + dh & cx + cy & cr + ds \end{pmatrix}_{2\times3}$$

Where, $R_1 = (a\ b)$ $stands\ for\ row$ 1 and $C^1 = \binom{W}{h}$ $stands\ for\ column$ 1 , $R_2 = (c\ d)$ $stands\ for\ row$ 2 and $C^2 = \binom{x}{y}$ $stands\ column$ 2, $C^3 = \binom{r}{s}$ $stands\ column$ 3

Example

1. Let A = $\begin{pmatrix} 2 & 0 & 8 \\ 3 & 5 & 1 \end{pmatrix}_{23}$ and B = $\begin{pmatrix} 2 & 3 \\ 0 & 5 \\ 8 & 1 \end{pmatrix}_{32}$ the find

   a. AB b. BA

2. Let $A = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 3 & -1 & 1 \end{pmatrix}$ $and\ B = \begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix}$ $be\ matrices\ determine$ AB?

Solution:

1. A = $\begin{pmatrix} 2 & 0 & 8 \\ 3 & 5 & 1 \end{pmatrix}_{23}$ and B = $\begin{pmatrix} 2 & 3 \\ 0 & 5 \\ 8 & 1 \end{pmatrix}_{32}$

   a. AB = ?

   Step 1: List out rows of A first, $R_1 = (2\ \ 0\ \ 8\ )$ and $R_2 = (3\ \ 5\ \ 1\ )$

   Step2 : list out columns of B : $C^1 = \begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix}$ and $C^2 = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}$

   Step3 : AB= $\begin{pmatrix} R_1C^1 & R_1C^2 \\ R_2C^1 & R_2C^2 \end{pmatrix} = \begin{pmatrix} (2\ \ 0\ \ 8\ )\begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix} & (2\ \ 0\ \ 8\ )\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \\ (3\ \ 5\ \ 1\ )\begin{pmatrix} 2 \\ 0 \\ 8 \end{pmatrix} & (3\ \ 5\ \ 1\ )\begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} \end{pmatrix}$

$$= \begin{pmatrix} 2\times2+0\times0+8\times8 & 2\times3+0\times5+8\times5 \\ 3\times2+5\times0+1\times8 & 3\times3+5\times5+1\times1 \end{pmatrix}$$

$$= \begin{pmatrix} 4+64 & 6+40 \\ 6+8 & 9+25+1 \end{pmatrix}$$

$$AB = \begin{pmatrix} 70 & 46 \\ 14 & 35 \end{pmatrix}$$

b. BA list out rows of B and columns of A

Rows of B ,$R_1 = (2\ \ 3\ ),\ R_2 = (0\ \ 5\ ), and\ R_3 = (8\ \ 1\ ),$

Columns of A, $C^1 = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $C^2 = \begin{pmatrix} 0 \\ 5 \end{pmatrix}$ and $C^3 = \begin{pmatrix} 8 \\ 1 \end{pmatrix}$

$$\text{BA}=\begin{pmatrix} R_1C^1 & R_1C^2 & R_1C^2 \\ R_2C^1 & R_2C^2 & R_2C^2 \\ R_3C^1 & R_3C^2 & R_3C^2 \end{pmatrix} = \begin{pmatrix} (2\ \ 3\ )\begin{pmatrix} 2 \\ 3 \end{pmatrix} & (2\ \ 3\ )\begin{pmatrix} 0 \\ 5 \end{pmatrix} & (2\ \ 3\ )\begin{pmatrix} 8 \\ 1 \end{pmatrix} \\ (0\ \ 5\ )\begin{pmatrix} 2 \\ 3 \end{pmatrix} & (0\ \ 5\ )\begin{pmatrix} 0 \\ 5 \end{pmatrix} & (0\ \ 5\ )\begin{pmatrix} 8 \\ 1 \end{pmatrix} \\ (8\ \ 1\ )\begin{pmatrix} 2 \\ 3 \end{pmatrix} & (8\ \ 1\ )\begin{pmatrix} 0 \\ 5 \end{pmatrix} & (8\ \ 1\ )\begin{pmatrix} 8 \\ 1 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 2\times 2+3\times 3 & 2\times 0+3\times 5 & 2\times 8+3\times 1 \\ 0\times 2+5\times 3 & 0\times 0+5\times 5 & 0\times 0+5\times 1 \\ 8\times 2+1\times 3 & 8\times 0+1\times 5 & 8\times 8+1\times 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4+9 & 0+15 & 16+3 \\ 2+15 & 0+25 & 0+5 \\ 16+3 & 0+5 & 64+1 \end{pmatrix}$$

$$\text{BA} \quad = \begin{pmatrix} 13 & 15 & 19 \\ 17 & 25 & 5 \\ 19 & 5 & 65 \end{pmatrix}$$

2. $A = \begin{pmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 3 & -1 & 1 \end{pmatrix}$ $and\ B = \begin{pmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 3 \end{pmatrix}$

a. $$\text{AB} = \begin{pmatrix} (1\ \ 2\ -3\ )\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} & (1\ \ 2\ -3\ )\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} & (1\ \ 2\ -3\ )\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \\ (5\ \ 0\ \ 2\ )\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} & (5\ \ 0\ \ 2)\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} & (5\ \ 0\ \ 2)\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \\ (3\ -1\ \ 1\ )\begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} & (3\ -1\ \ 1\ )\begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix} & (3\ -1\ \ 1\ )\begin{pmatrix} 2 \\ 5 \\ 3 \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 3+8-6 & -1+4+0 & 2+10-9 \\ 15+0+4 & -5+0+0 & 25+0+6 \\ 12-4+2 & -3-2+0 & 6-5+3 \end{pmatrix}$$

$$\text{AB} \quad = \begin{pmatrix} 5 & 3 & 3 \\ 19 & -5 & 31 \\ 10 & -5 & 4 \end{pmatrix}$$

Properties of Multiplication of Matrices

Let A , B and C be matrices in real numbers and $r \in R$, then

- $A(B + C) = AB + AC \ldots left\ distributive\ property$
- $(A+B)C = AC + BC \quad \ldots right\ distributive\ property$
- $A(BC) = (AB)C \ldots\ldots\ldots \quad associative\ property$
- $r(AB) = (rA)B = A(rB) \ldots\ldots scalar\ multiplication$

NB: the product of two non-zero matrices can be a zero matrix

Example:

1 let A = $\begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$ $and\ B = \begin{pmatrix} 3 & -3 \\ -2 & 2 \end{pmatrix}$ then show that $AB = 0$

Solution: AB= $\begin{pmatrix} (2 \quad 3)\begin{pmatrix} 3 \\ -2 \end{pmatrix} & (2 \quad 3)\begin{pmatrix} -3 \\ 2 \end{pmatrix} \\ (2 \quad 3)\begin{pmatrix} 3 \\ -2 \end{pmatrix} & (2 \quad 3)\begin{pmatrix} -3 \\ 2 \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 6-6 & -6+6 \\ 6-6 & -6+6 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

2.4. Transpose of matrices and its property

Definition:

For any matrix A, the matrix obtained from A by interchanging its rows and columns is the transpose of A denoted by A^t

i.e. if A= $(a_{ij})_{mn}$ the $A^t = (a_{ji})_{nm}$

Example:

1. Find the transpose of the following matrices.

   a. A = $\begin{pmatrix} 2 & 3 & 5 \\ 8 & 9 & 2 \\ 0 & 4 & 1 \end{pmatrix}$ b. B = $(6 \quad 4 \quad 8 \quad 1)$ c. C = $\begin{pmatrix} 7 \\ 2 \\ 0 \end{pmatrix}$ d. D= (9)

2. Let x and y be real numbers, A= $\begin{pmatrix} 2 & x+1 & 5 \\ 8 & 9 & 2 \\ 0 & 4 & 1 \end{pmatrix}$ $and \quad B = \begin{pmatrix} 2 & 8 & 0 \\ 2x-y & 9 & x+y \\ 0 & 2 & 1 \end{pmatrix}$

   Then find the value of x and y when $A^t = B$

3. If A = $\begin{pmatrix} -1 & 2 & 0 \\ -1 & -2 & 1 \\ 3 & 1 & 4 \end{pmatrix}$, then find matrix B such that $B - 2A^t = A$

Solution:

1. a. A = $\begin{pmatrix} 2 & 3 & 5 \\ 8 & 9 & 2 \\ 0 & 4 & 1 \end{pmatrix} \Rightarrow A^t = \begin{pmatrix} 2 & 8 & 0 \\ 3 & 9 & 4 \\ 5 & 2 & 1 \end{pmatrix}$ b. $B = (6 \quad 4 \quad 8 \quad 1) \Rightarrow B^t = \begin{pmatrix} 6 \\ 4 \\ 8 \\ 1 \end{pmatrix}$

c. $C = \begin{pmatrix} 7 \\ 2 \\ 0 \end{pmatrix} \Longrightarrow C^t = \begin{pmatrix} 7 & 2 & 0 \end{pmatrix}$ d. D= $(9) \Longrightarrow D^t = (9)$

2. A= $\begin{pmatrix} 2 & x+1 & 5 \\ 8 & 9 & 2 \\ 0 & 4 & 1 \end{pmatrix} \Longrightarrow A^t = \begin{pmatrix} 2 & 8 & 0 \\ x+1 & 9 & 4 \\ 5 & 2 & 1 \end{pmatrix} \Longrightarrow A^t = B$