Lesson 2: Special Types of Matrices

I. Symmetric matrix
ii. skew-symmetric matrix

i. Symmetric matrix

Definition: any square matrix A is said to be symmetric if it is equal to its transpose.
i.e. $A = A^t$

Example:

1. Show that $A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ is symmetric?
2. Which of the following matrices is /are symmetric?
   A= $\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & h \end{pmatrix}$ B= $\begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & 4 \\ -2 & 4 & 3 \end{pmatrix}$ C= $\begin{pmatrix} 2 & 3 \\ 3 & 1 \\ 1 & 4 \end{pmatrix}$
3. Let A= $\begin{pmatrix} 3 & r-2s & x+3 \\ 2r+3s & 2x+3r & 3r+5 \\ 5x-7 & 2s-6 & 3r \end{pmatrix}$ be a symmetric matrix then find x, r and s?

Solution:

1. $if\ A = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 3 \\ 2 & 3 & 1 \end{pmatrix}\ then\ A^t = \begin{pmatrix} 1 & -1 & 2 \\ -1 & 0 & 3 \\ 2 & 3 & 1 \end{pmatrix} \Longrightarrow A = A^t$
   ,$\therefore A\ is\ a\ symmetric\ matrix$
2. . A= $\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & h \end{pmatrix}, then\ A^t = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & h \end{pmatrix} \Longrightarrow A = A^t \therefore A\ is\ symmetric$
   .B= $\begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & 4 \\ -2 & 4 & 3 \end{pmatrix}, then\ B^t = \begin{pmatrix} 1 & 2 & -2 \\ 2 & -2 & 4 \\ -2 & 4 & 3 \end{pmatrix} \Longrightarrow B = B^t$

   ,therefore B is symmetric

   .C= $\begin{pmatrix} 2 & 3 \\ 3 & 1 \\ 1 & 4 \end{pmatrix}$ and $C^t = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 4 \end{pmatrix} \Longrightarrow C \neq C^t\ \therefore c\ is\ not\ symmetric$
3. A= $\begin{pmatrix} 3 & r-2s & x+3 \\ 2r+3s & 2x+3r & 3r+5 \\ 5x-7 & 2s-6 & 3r \end{pmatrix}$
   If A is symmetric then $A^t = A$

   $. \Longrightarrow \begin{pmatrix} 3 & 2r+3s & 5x-7 \\ r-2s & 2x+3r & 2s-6 \\ x+3 & 3r+5 & 3r \end{pmatrix} = \begin{pmatrix} 3 & r-2s & x+3 \\ 2r+3s & 2x+3r & 3r+5 \\ 5x-7 & 2s-6 & 3r \end{pmatrix}$

   $, \Longrightarrow x+3 = 5x-7 \Longrightarrow 4x = 10 \Longrightarrow x = \frac{5}{2}$

$,\Longrightarrow 2r+3s = r-2s \quad and \quad 2s-6 = 3r+5$

$$,\begin{cases} r+5s=0 \\ 2s-3r=11 \end{cases} \quad \Longrightarrow r=-5s \; and \; substite \;\; r \; by-5s$$

$$,2s-3(-5s)=11 \Longrightarrow s=\frac{11}{17} \; and \; r=-5s=-\frac{55}{17}$$

Therefore the values of x, r and s are $\frac{5}{2}$, $-\frac{55}{17}$ $and\,\frac{11}{17}$ $respectively$

ii. **Skew Symmetric Matrix**

Definition: any square matrix A is said to be skew-symmetric if $A=-A^t$ or $A^t+A=O$

Example :

1. Which of the following matrices are skew-symmetric?

   A= $\begin{pmatrix} 0 & -1 & 4 \\ 1 & 0 & 7 \\ -4 & -7 & 0 \end{pmatrix}$ B= $\begin{pmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{pmatrix}$ C= $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 4 \\ 3 & 4 & 6 \end{pmatrix}$
2. For any square matrix A Show that $A-A^t$ is skew- symmetric?
3. If A = $\begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$ then show that $AA^t=A^tA=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$
4. Let A = $\begin{pmatrix} 1 & x & 2 \\ 0 & 1 & 0 \\ y & z & 5 \end{pmatrix}$ find x, y , z such that

   i.A is symmetric

   ii. A is triangular matrix
5. Let A= $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ be a matrix , then find

   a. $B=\frac{A+A^t}{2}$ and show B is symmetric

   b. $D=\frac{A-A^t}{2}$ and show D is skew-symmetric

   c. $show \; that \; B+D \; = A$

Solution

1. A= $\begin{pmatrix} 0 & -1 & 4 \\ 1 & 0 & 7 \\ -4 & -7 & 0 \end{pmatrix}$ $and \; A^t=\begin{pmatrix} 0 & 1 & -4 \\ -1 & 0 & -7 \\ 4 & 7 & 0 \end{pmatrix}$

   $$, A+A^t=\begin{pmatrix} 0 & -1 & 4 \\ 1 & 0 & 7 \\ -4 & -7 & 0 \end{pmatrix}+\begin{pmatrix} 0 & 1 & -4 \\ -1 & 0 & -7 \\ 4 & 7 & 0 \end{pmatrix}=\begin{pmatrix} 0+0 & -1+1 & 4+(-4) \\ 1+(-1) & 0+0 & 7+(-7) \\ -4+4 & -7+7 & 0+0 \end{pmatrix}$$

   $$A+A^t \;=\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

   Therefore A is a skew- symmetric matrix.

   B = $\begin{pmatrix} 0 & a & b \\ -a & 0 & -c \\ -b & c & 0 \end{pmatrix}$ $and \; B^t=\begin{pmatrix} 0 & -a & -b \\ a & 0 & c \\ b & -c & 0 \end{pmatrix}$

$$B+B^t=\begin{pmatrix}0 & a & b\\ -a & 0 & -c\\ -b & c & 0\end{pmatrix}+\begin{pmatrix}0 & -a & -b\\ a & 0 & c\\ b & -c & 0\end{pmatrix}=\begin{pmatrix}0+0 & a+(-a) & b+(-b)\\ -a+a & 0+0 & -c+c\\ -b+b & c+(-c) & 0+0\end{pmatrix}$$

$$B+B^t=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

Therefore B is skew-symmetric

$$C=\begin{pmatrix}1 & 2 & 3\\ 2 & 5 & 4\\ 3 & 4 & 6\end{pmatrix} and\ c^t=\begin{pmatrix}1 & 2 & 3\\ 2 & 5 & 4\\ 3 & 4 & 6\end{pmatrix}\Longrightarrow C+C^t=\begin{pmatrix}1+1 & 2+2 & 3+3\\ 2+2 & 5+5 & 4+4\\ 3+3 & 4+4 & 6+6\end{pmatrix}$$

, $\Longrightarrow C+C^t=\begin{pmatrix}2 & 4 & 6\\ 4 & 10 & 8\\ 6 & 8 & 12\end{pmatrix}$ therefore C is not skew-symmetric since $C+C^t\neq O$

2. For any square matrix A Show that $A-A^t$ is skew- symmetric?

Let A be a square matrix then $A-A^t+(A-A^t)^t=A-A^t+A^t-A=O$

It Shows that $A-A^t\ is\ skew-symmetric$

3. A $=\begin{pmatrix}cos\theta & -sin\theta\\ sin\theta & cos\theta\end{pmatrix}$ and $AA^t=\begin{pmatrix}cos\theta & -sin\theta\\ sin\theta & cos\theta\end{pmatrix}\begin{pmatrix}cos\theta & sin\theta\\ -sin\theta & cos\theta\end{pmatrix}$

=

$$AA^t=\begin{pmatrix}(cos\theta\quad -sin\theta)\begin{pmatrix}cos\theta\\ -sin\theta\end{pmatrix} & (cos\theta\quad -sin\theta)\begin{pmatrix}sin\theta\\ cos\theta\end{pmatrix}\\ (sin\theta\quad cos\theta)\begin{pmatrix}cos\theta\\ -sin\theta\end{pmatrix} & (sin\theta\quad cos\theta)\begin{pmatrix}sin\theta\\ cos\theta\end{pmatrix}\end{pmatrix}$$

$$=\begin{pmatrix}cos\theta cos\theta+sin\theta sin\theta & cos\theta sin\theta-cos\theta sin\theta\\ sin\theta cos\theta-sin\theta cos\theta & sin\theta sin\theta+co\theta cos\theta\end{pmatrix}$$

$=\begin{pmatrix}\cos^2\theta+\sin^2\theta & 0\\ 0 & \sin^2\theta+\cos^2\theta\end{pmatrix}$, where $\cos^2\theta+\sin^2\theta=1$

, $AA^t=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$ is a $2\times 2 identity\ matrix$

4. $A=\begin{pmatrix}1 & x & 2\\ 0 & 1 & 0\\ y & z & 5\end{pmatrix}$ i. if A is symmetric $A=A^t$

Then $\begin{pmatrix}1 & x & 2\\ 0 & 1 & 0\\ y & z & 5\end{pmatrix}=\begin{pmatrix}1 & 0 & y\\ x & 1 & 5\\ 2 & 0 & 5\end{pmatrix}\Longrightarrow x=0, y=2\ and\ z=0$

ii.if A is triangular

a. if A is upper triangular, then y = z =0 and $x\in R$
b. if A is lower triangular, then x = 0 and $y,z\in R$
c. if A is both, x = y = z = 0.

5. A= $\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}$

a. $B=\frac{A+A^t}{2}$ and show B is symmetric

$$B=\frac{1}{2}(A+A^t)=\frac{1}{2}\left(\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}+\begin{pmatrix}1 & 4 & 7\\ 2 & 5 & 8\\ 3 & 6 & 9\end{pmatrix}\right)=\frac{1}{2}\begin{pmatrix}2 & 6 & 10\\ 6 & 10 & 14\\ 10 & 14 & 18\end{pmatrix}=\begin{pmatrix}1 & 3 & 5\\ 3 & 5 & 7\\ 5 & 7 & 9\end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{pmatrix} \text{ is symmetric}$$

b. $D = \frac{A-A^t}{2}$ and show D is skew-symmetric

D = $\frac{1}{2}(A - A^t)$

$$= \frac{1}{2}\left(\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix} - \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}\right) = \frac{1}{2}\begin{pmatrix} 0 & -2 & -4 \\ 2 & 0 & -2 \\ 4 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix}$$

$$D = \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} \text{ is a skew- symmetric matrix}$$

c. $show\ that\ B + D\ = A$

$$B + D = \begin{pmatrix} 1 & 3 & 5 \\ 3 & 5 & 7 \\ 5 & 7 & 9 \end{pmatrix} + \begin{pmatrix} 0 & -1 & -2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$$

Note (generalized from the above)

Any square matrix can be expressed as the sum of symmetric and skew-symmetric matrices

Elementary Operations on Matrices

Definition

An elementary operation of matrices is a simple operation made on rows or columns which aims transforming any matrix to row or column equivalent matrix.

Some Elementary operations

- Elementary row operations
- Elementary column operations

i. Elementary row operations:

- ❖ Operation Methods

Swapping:- interchanging rows of a matrix $(R_i \leftrightarrow R_j)$

Re-scaling:- multiplying a row of a matrix by anon zero constant $(R_i \longrightarrow kR_i)$

Pivoting :- adding constant multiple of one row of a matrix on to another row.$(R_j \longrightarrow R_j + kR_i)$

Example

1. Let A $=\begin{pmatrix}5 & 2 & 3\\ 1 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}$ be a matrix the transform A to its row equivalent matrix using elementary row operations

Solution:

Step 1: row interchange (swapping)

$$,R_1 \leftrightarrow R_2 \begin{pmatrix}1 & 5 & 6\\ 5 & 2 & 3\\ 7 & 8 & 9\end{pmatrix}$$

Step 2 : re-scaling row 1 to eliminate row 2 and 3

a. To eliminate first entry of row 2 multiply row one by -5
   $,R_1 \longrightarrow -5R_1 \implies R_1 = (-5 \ -25 \ -30)$
b. To eliminate first entry of row three multiply row one by -7
   $,R_1 \longrightarrow -7R_1 \implies R_1 = (-7 \ -35 \ -42\ )$

Step 3: pivoting (adding the rescaled rows to row 2 and row 3 respectively)

$$,R_2 \longrightarrow R_2 + -5R_1 \begin{pmatrix}1 & 5 & 6\\ 5+(-5) & 2+(-25) & 3+(-30)\\ 7 & 8 & 9\end{pmatrix} = \begin{pmatrix}1 & 5 & 6\\ 0 & -23 & -27\\ 7 & 8 & 9\end{pmatrix}$$

$$,\ R_3 \longrightarrow R_3 + -7R_1 \begin{pmatrix}1 & 5 & 6\\ 0 & -23 & -27\\ 7+(-7) & 8+(-35) & 9+(-42)\end{pmatrix} = \begin{pmatrix}1 & 5 & 6\\ 0 & -23 & -27\\ 0 & -27 & -31\end{pmatrix}$$

Step 4: again rescale row two to eliminate second entry (-27) of row three

$$,\ R_2 \longrightarrow -\frac{27}{23}R_2 = (0 \quad 27 \quad \frac{729}{23})$$

Step 5: pivoting (adding rescaled row to third row to eliminate -27

$$,\ R_3 \longrightarrow R_3 + -\frac{27}{23}R_1 \begin{pmatrix}1 & 2 & 6\\ 0 & -23 & -27\\ 0+0 & -27+27 & -31+\frac{729}{23}\end{pmatrix} = \begin{pmatrix}1 & 5 & 6\\ 0 & -23 & -27\\ 0 & 0 & \frac{16}{23}\end{pmatrix}$$

The elementary operation we do transforms matrix A to its row equivalent matrix

$$,\begin{pmatrix}1 & 5 & 6\\ 0 & -23 & -27\\ 0 & 0 & \frac{16}{23}\end{pmatrix} \textit{ is the row equivalent matrix for } A = \begin{pmatrix}5 & 2 & 3\\ 1 & 5 & 6\\ 7 & 8 & 9\end{pmatrix}$$

ii. Elementary column operations:

❖ Operation Methods

Swapping: - interchanging column of a matrix $(C_i \leftrightarrow C_j)$

Re-scaling:- multiplying a column of a matrix by anon zero constant $(C_i \longrightarrow kC_i)$

Pivoting:- adding constant multiple of one column of a matrix on to another column

$.(C_j \longrightarrow C_j + kC_i)$

Example:

2. 1. Let $A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 1 & 2 \\ 2 & 3 & 4 \end{pmatrix}$ be a matrix the transform A to its equivalent matrix using elementary column operations

Solution:

Step1: swapping

$$, C_1 \leftrightarrow C_2 \begin{pmatrix} 1 & 0 & 3 \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{pmatrix}$$

Step2: rescaling $C_1 \longrightarrow -3C_1$ $to\ eliminate\ first\ enry\ on\ column\ three$

$$, C_1 \longrightarrow -3C_1 \begin{pmatrix} -3 \\ -3 \\ -9 \end{pmatrix}$$

Step 3 : pivoting (eliminate first entry of column three by adding rescaled column

$$,C_3 \longrightarrow -3C_1 + C_3 \begin{pmatrix} 1 & 0 & -3+3 \\ 1 & 1 & -3+2 \\ 3 & 2 & -9+4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 3 & 2 & -5 \end{pmatrix}$$

Step 4: pivoting to eliminate second entry of third column by direct addition with column two

$$,C_3 \longrightarrow C_3 + C_2 \begin{pmatrix} 1 & 0 & 0+0 \\ 1 & 1 & -1+1 \\ 3 & 2 & -5+2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & -3 \end{pmatrix}$$

$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 3 & 2 & -3 \end{pmatrix}$ $is\ column\ reduced\ form\ of\ A$

Echelon form of a matrix

Definition:

A matrix is said to be in **row echelon** form if:

- A zero row(if there is) comes at the bottom
- The first non-zero entry in each non-zero row is 1
- The number of zero entries down the row is increasing.

Example:

1. Identify which of the following matrices are in echelon form

$$A=\begin{pmatrix}1 & 5 & 8\\ 0 & 1 & 3\\ 0 & 0 & 0\end{pmatrix} \quad B=\begin{pmatrix}1 & 3 & 4\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} \quad C=\begin{pmatrix}1 & 6\\ 0 & 1\\ 0 & 0\end{pmatrix}$$

$$D=\begin{pmatrix}1 & 6\\ 0 & 0\end{pmatrix} \quad E=\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix} \quad G=\begin{pmatrix}1 & 2 & 0\\ 0 & 0 & 2\\ 0 & 0 & 0\end{pmatrix}$$

Solution:

- Matrices A , C , D and E are in row echelon form.
- matrix B is not in echelon form because the number of zeros on second row is greater than the number of zeros on third row.
- Matrix G is not an echelon matrix because the first non- zero entry on the second row is not 1

Definition:

A matrix is said to be row reduced echelon (RREF) if and only if

- It is in echelon form
- The first non-zero entry in each non-zero row is the only non-zero entry in its column.

Example:

1. Identify which of the following matrices are row reduced echelon matrices

$$A=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{pmatrix} \qquad B=\begin{pmatrix}1 & 2 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{pmatrix} \qquad C=\begin{pmatrix}1 & 2 & 0\\ 0 & 0 & 1\end{pmatrix}$$

Solution:

- Matrices A and C are in RREF
- Matrix B is not in RREF because the first non-zero entry on row two is not the only nonzero entry a cross its column.
-

2. Reduce the following matrices in to row echelon form.

a. $\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 1 & 1 & 7\end{pmatrix}$ b. $\begin{pmatrix}3 & 2 & 1 & 2\\ 1 & 3 & 1 & 2\\ 2 & 1 & 4 & 3\end{pmatrix}$

Solution

$$\text{a.} \Longrightarrow R_2 \longrightarrow R_2 + -4R_1 \begin{pmatrix} 1 & 2 & 3 \\ 4+(-4) & 5+(-8) & 6+(-12) \\ 1 & 1 & 7 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 1 & 1 & 7 \end{pmatrix}$$

$$, \Longrightarrow R_3 \longrightarrow R_3 + -R_1 \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 1+(-1) & 1+(-2) & 7+(-3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -1 & 4 \end{pmatrix}$$

$$, \Longrightarrow R_2 \longrightarrow \frac{R_2}{-3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & 4 \end{pmatrix}$$

$$, \Longrightarrow R_3 \longrightarrow R_3 + R_2 \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -1+1 & 4+2 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{pmatrix}$$

$$, \Longrightarrow R_3 \longrightarrow \frac{R_3}{6} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & \frac{6}{6} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$, \therefore \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \text{ is the row echelon form}$$

$$b. \begin{pmatrix} 3 & 2 & 1 & 2 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{pmatrix} \Longrightarrow R_1 \leftrightarrow R_2 \begin{pmatrix} 1 & 3 & 1 & 2 \\ 3 & 2 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{pmatrix} \Rightarrow R_2 \longrightarrow R_2 + -3R_1 \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -7 & -2 & -4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$$

$$, \Rightarrow R_3 \longrightarrow R_3 + -2R_1 \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -7 & -2 & -4 \\ 0 & -5 & 2 & -1 \end{pmatrix} \Rightarrow R_2 \longrightarrow -\frac{R_2}{7} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & \frac{2}{7} & \frac{4}{7} \\ 0 & -5 & 2 & -1 \end{pmatrix}$$

$$, \Rightarrow R_3 \longrightarrow R_3 + 5R_2 \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & \frac{2}{7} & \frac{4}{7} \\ 0 & 0 & \frac{24}{7} & \frac{13}{7} \end{pmatrix} \Rightarrow R_3 \longrightarrow \frac{7}{24} R_3 \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & \frac{2}{7} & \frac{4}{7} \\ 0 & 0 & 1 & \frac{13}{24} \end{pmatrix}$$

$$\therefore \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & \frac{2}{7} & \frac{4}{7} \\ 0 & 0 & 1 & \frac{13}{24} \end{pmatrix} \; is\ the\ echelon\ form$$

3. Reduce each of the following matrices in to row reduced echelon form(RREF)

a. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 1 & 7 \end{pmatrix}$ b. $\begin{pmatrix} 3 & 2 & 1 & 2 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

solution:

a. $\Longrightarrow R_2 \longrightarrow R_2 + -4R_1 \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 1 & 1 & 7 \end{pmatrix} \Longrightarrow R_3 \longrightarrow R_3 + -R_1 \begin{pmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -1 & 4 \end{pmatrix}$

$, \Longrightarrow R_2 \longrightarrow \frac{R_2}{-3} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & -1 & 4 \end{pmatrix} \Longrightarrow R_3 \longrightarrow R_3 + R_2 \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{pmatrix}$

$, , \Longrightarrow R_3 \longrightarrow \frac{R_3}{6} \begin{pmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix} \Longrightarrow R_1 \longrightarrow R_1 + -2R_2 \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$

$, \Longrightarrow \frac{R_1 \longrightarrow R_1 + R_3}{R_2 \longrightarrow R_2 + -2R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$, \therefore \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} is\ the\ row\ reduced\ echelon\ form$

b. $\begin{pmatrix} 3 & 2 & 1 & 2 \\ 1 & 3 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$\Longrightarrow R_1 \leftrightarrow R_2 \begin{pmatrix} 1 & 3 & 1 & 2 \\ 3 & 2 & 1 & 2 \\ 2 & 1 & 4 & 3 \end{pmatrix} \Rightarrow R_2 \longrightarrow R_2 + -3R_1 \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -7 & -2 & -4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$

$\Rightarrow R_3 \longrightarrow R_3 + -2R_1 \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & -7 & -2 & -4 \\ 0 & -5 & 2 & -1 \end{pmatrix} \Rightarrow R_2 \longrightarrow -\frac{R_2}{7} \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & \frac{2}{7} & \frac{4}{7} \\ 0 & -5 & 2 & -1 \end{pmatrix}$

$, \Rightarrow R_3 \longrightarrow R_3 + 5R_2 \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & \frac{2}{7} & \frac{4}{7} \\ 0 & 0 & \frac{24}{7} & \frac{13}{7} \end{pmatrix} \Rightarrow R_3 \longrightarrow \frac{7}{24} R_3 \begin{pmatrix} 1 & 3 & 1 & 2 \\ 0 & 1 & \frac{2}{7} & \frac{4}{7} \\ 0 & 0 & 1 & \frac{13}{24} \end{pmatrix}$

$\Longrightarrow R_1 \longrightarrow R_{1+} + -3R_2 \begin{pmatrix} 1 & 0 & \frac{1}{7} & \frac{2}{7} \\ 0 & 1 & \frac{2}{7} & \frac{4}{7} \\ 0 & 0 & 1 & \frac{13}{24} \end{pmatrix} \Rightarrow \frac{R_1 \longrightarrow R_1 + -\frac{1}{7}R_3}{R_2 \longrightarrow R_2 + -\frac{2}{7}R_3} \begin{pmatrix} 1 & 0 & 0 & \frac{35}{168} \\ 0 & 1 & 0 & \frac{35}{84} \\ 0 & 0 & 1 & \frac{13}{24} \end{pmatrix}$

$.\therefore \begin{pmatrix} 1 & 0 & 0 & \frac{35}{168} \\ 0 & 1 & 0 & \frac{35}{84} \\ 0 & 0 & 1 & \frac{13}{24} \end{pmatrix}$ is the row reduced echelon form.