Lesson 3 : System of Linear Equations

Definition:

Let $a_{ij}, b_j \in \mathcal{R}\ for\ i = 1,2,3, \ldots m\ and\ j = 1,2,3, \ldots, n$. A finite set of linear equations;

$$\begin{matrix} a_{11}x_1 + & a_{12}x_2 + \cdots + & a_{1n}x_n = b_1 \\ a_{21}x_1 + & a_{22}x_2 + \cdots + & a_{2n}x_n = b_2 \\ \vdots \quad \cdots & \vdots \quad \cdots & \vdots \quad \cdots \quad \vdots \\ a_{m1}x_m + & a_{m2}x_2 + \cdots + & a_{mn}x_n = b_m \end{matrix}$$

is said to be a system of linear equations with n variables i.e. $x_1, x_2, x_3, \ldots, x_n$

Example:

1. Determine the number of equations and variables that the following system of linear equations do have?

A

$$\begin{aligned} 3x + 5y + z + r &= 0 \\ x - 5y + 3z + 8r &= 0 \\ 5x + y - 5z + 6r &= 0 \\ 7x + 2y - 3z - 9r &= 0 \end{aligned}$$

B.

$$\begin{aligned} x - 5y + z &= 6 \\ 5x + 3y - 6z &= 0 \\ 2x + y - 9z &= 2 \\ 7x + 8y + z &= 6 \end{aligned}$$

Solution

A. The system has
- Four equations
- Four variable
- It is a homogenous equation

B. the system has
- four equations
- three variables
- it is a non-homogenous equation

❖ **NB** In a Linear System

$$\begin{matrix} a_{11}x_1 + & a_{12}x_2 + \cdots + & a_{1n}x_n = b_1 \\ a_{21}x_1 + & a_{22}x_2 + \cdots + & a_{2n}x_n = b_2 \\ \vdots \quad \cdots & \vdots \quad \cdots & \vdots \quad \cdots \quad \vdots \\ a_{m1}x_m + & a_{m2}x_2 + \cdots + & a_{mn}x_n = b_m \end{matrix}$$

where, $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ $are\ all\ zero\ we\ call\ it\ homogenous\ system\ of linear\ equations$

and $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ are non- zero we call it non-homogenous system of linear equations.

- The above system of linear equation can also be written in the form, $AX = B$
  - X= $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ and B= $\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$
  - A= $\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$ is the coefficient matrix.
  - $(A|B) = \left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$ is the augmented matrix

Example

1.
2. Identify
   a. coefficient matrix
   b. augmented matrix for the following equation

$$\begin{aligned} x - 5y + z &= 6 \\ 5x + 3y - 6z &= 0 \\ 2x + y - 9z &= 2 \\ 7x + 8y + z &= 6 \end{aligned}$$

Solution

a. the coefficient matrix is A= $\begin{pmatrix} 1 & -5 & 1 \\ 5 & 3 & -6 \\ 2 & 1 & -9 \\ 7 & 8 & 1 \end{pmatrix}$

b. the Augmented matrix is $(A|B) = \left(\begin{array}{ccc|c} 1 & -5 & 1 & 6 \\ 5 & 3 & -6 & 0 \\ 2 & 1 & -9 & 2 \\ 7 & 8 & 1 & 6 \end{array}\right)$

1.8.1. Solutions of System of Linear Equations

There are different methods of finding solutions

- Elimination
- Gauss- method
- Gauss-Jordan method

1. Elimination method

- This method is effective on 3-4 number of equations and variables.

Example

Solve the system $\begin{cases} x+y+z=3 \\ x-2y+z=1 \\ 3x+2y-3z=4 \end{cases}$

Solution

Eliminate the number of equations

$\begin{cases} x+y+z=3 \\ x-2y+z=1 \end{cases}$

$\Rightarrow x=3-y-z$

$\Rightarrow 3-y-z-2y+z=1$

$\Rightarrow -3y=-2$

$\Rightarrow y=\frac{2}{3}$

$\begin{cases} x-2y+z=1 \\ 3x+2y-3z=4 \end{cases}$

$x=1+2y-z$

$3(1+2y-z)+2y-3z=4$

$3+6y-3z+2y-3z=4$

$8y-6z=1$

$\Rightarrow z=\frac{8y-1}{6}$

$\Rightarrow z=\frac{8\left(\frac{2}{3}\right)-1}{6}$

$\Rightarrow z=\frac{13}{18}$

$\begin{cases} x+y+z=3 \\ 3x+2y-3z=4 \end{cases}$

$x=3-y-z$

$x=3-\frac{2}{3}-\frac{13}{18}$

$x=\frac{54-25}{18}$

$x=\frac{29}{18}$

Solution of the system is a row matrix $\begin{pmatrix} x & y & z \end{pmatrix}=\begin{pmatrix} \frac{29}{18} & \frac{2}{3} & \frac{13}{18} \end{pmatrix}$

2. Gaussian Method

Gauss used elementary row / column operations on augmented matrix

- **Swapping**:- interchanging rows of a matrix $(R_i \leftrightarrow R_j)$
- **Re-scaling:-** multiplying a row of a matrix by anon zero constant $(R_i \longrightarrow kR_i)$
- **Pivoting :-** adding constant multiple of one row of a matrix on to another row.$(R_j \longrightarrow R_j + kR_i)$

Example1

1. Solve the system using Gauss method $\begin{cases} x+y+z=3 \\ x-2y+z=1 \\ 3x+2y-3z=4 \end{cases}$

Solution

Transform in matrix form;

$A=\begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 3 & 2 & -3 \end{pmatrix}, X=\begin{pmatrix} x \\ y \\ z \end{pmatrix}, B=\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix}$ where X is the solution vector

Step1: Augmented matrix $A|B=\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -2 & 1 & 1 \\ 3 & 2 & -3 & 4 \end{array}\right)$

Step 2: change A|B into echelon form

$$,R_2 \longrightarrow -R_1 + R_2 \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 3 & 2 & -3 & 4 \end{pmatrix} \Longrightarrow, R_3 \longrightarrow -3R_1 + R_3 \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 0 & -1 & -6 & -5 \end{pmatrix}$$

$$\Rightarrow R_3 \longrightarrow -\frac{1}{3}R_2 + R_3 \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & -6 & -\frac{13}{3} \end{pmatrix} \Rightarrow \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & -6 & -\frac{13}{3} \end{pmatrix} is\ the\ echelon\ form$$

Using back substitution $\begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 0 & 0 & -6 & -\frac{13}{3} \end{pmatrix} \Longrightarrow \begin{matrix} x + y + z = 3 \\ -3y = -2 \\ -6z = -\frac{13}{3} \end{matrix}$

$$, \Rightarrow z = -\frac{13}{3(-6)} = \frac{13}{18} \quad , -3y = -2 \Longrightarrow y = \frac{2}{3} \quad and\ x = 3 - y - z = 3 - \frac{2}{3} - \frac{13}{18} = \frac{29}{18}$$

Therefore the solution is $= \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{29}{18} \\ \frac{2}{3} \\ \frac{13}{18} \end{pmatrix}$ or $(x \quad y \quad z) = \begin{pmatrix} \frac{29}{18} & \frac{2}{3} & \frac{13}{18} \end{pmatrix}$

3. Gauss – Jordan method

- This method uses the method of row reduced echelon (RREF)
- Eliminate until coefficient matrix is changed into an identity matrix

Example1:

1. Solve the system using Gauss-Jordan method $\begin{cases} x + y + z = 3 \\ x - 2y + z = 1 \\ 3x + 2y - 3z = 4 \end{cases}$

step 1 Augmented matrix $A|B = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -2 & 1 & 1 \\ 3 & 2 & -3 & 4 \end{array}\right)$

Step 2 : reduce rows;

$$R_2 \longrightarrow -R_1 + R_2 \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 3 & 2 & -3 & 4 \end{pmatrix} \Longrightarrow, R_3 \longrightarrow -3R_1 + R_3 \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & -3 & 0 & -2 \\ 0 & -1 & -6 & -5 \end{pmatrix}$$

$$\Longrightarrow R_2 \longrightarrow -\frac{1}{3}R_1 \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & -1 & -6 & -5 \end{pmatrix} \Rightarrow R_3 \longrightarrow R_2 + R_3 \begin{pmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & -6 & -\frac{13}{3} \end{pmatrix}$$

$$\Longrightarrow R_3 \longrightarrow R_{\frac{3}{-6}} \begin{pmatrix} 1 & 1 & 1 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{13}{18} \end{pmatrix} \Longrightarrow R_1 \longrightarrow -R_2 + R_1 \begin{pmatrix} 1 & 0 & 1 & \frac{7}{3} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{13}{18} \end{pmatrix}$$

$$\Longrightarrow R_1 \longrightarrow -R_3 + R_1 \begin{pmatrix} 1 & 0 & 0 & \frac{29}{18} \\ 0 & 1 & 0 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{13}{18} \end{pmatrix}$$ here the coefficient matrix is transformed in to identity

Then $\begin{matrix} x = \frac{29}{18} \\ y = \frac{2}{3} \\ z = \frac{13}{18} \end{matrix} \Longrightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{29}{18} \\ \frac{2}{3} \\ \frac{13}{18} \end{pmatrix}$

3.8.2. Different Solutions of System of Linear Equations

There are three solution types

- Unique (one) solution (consistent)
- Many solution (dependent, consistent)
- No solution (inconsistent)

1. Unique (one) solution

- A linear system has unique solution if and only if it has only one solution.

Example:

a. Solve $\begin{cases} 3x + 4y = 6 \\ x - 2y = 1 \end{cases}$

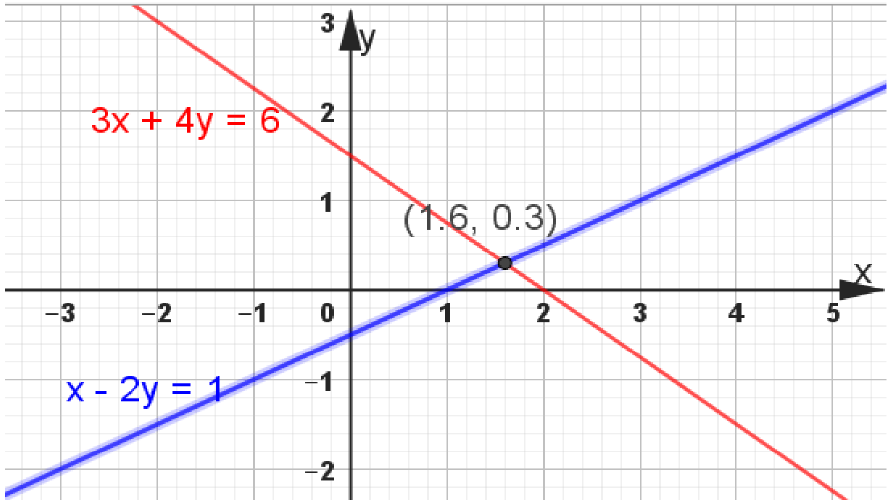
Solution:

$let\ l_1: 3x + 4y = 6\ and\ l_2: x - 2y = 1\ be\ two\ lines\ , if\ l_1\ and l_2\ intersect\ at\ a\ point\ then$ the system has one solution.

Find intersection point $x = 1 + 2y\ and\ 3(1 + 2y) + 4y = 6 \Longrightarrow 10y = 3 \Longrightarrow y = \frac{3}{10}$

And $x = 1 + 2\left(\frac{3}{10}\right) = \frac{16}{10} = \frac{8}{5}$ the intersection point is at $(x, y) = (\frac{8}{5}, \frac{3}{10})$

:



b. Solve $\begin{matrix} 3x+2y-z=6 \\ x-2y+3z=3 \\ 2x+y+z=4 \end{matrix}$ using gauss method.

Solution

Let A= $\begin{pmatrix} 3 & 2 & -1 \\ 1 & -2 & 3 \\ 2 & 1 & 1 \end{pmatrix}$ be the coefficient matrix and A|B= $\begin{pmatrix} 3 & 2 & -1 & 6 \\ 1 & -2 & 3 & 3 \\ 2 & 1 & 1 & 4 \end{pmatrix}$ be augmented matrix

$$\Rightarrow R_1 \leftrightarrow R_2 \begin{pmatrix} 1 & -2 & 3 & 3 \\ 3 & 2 & -1 & 6 \\ 2 & 1 & 1 & 4 \end{pmatrix} \xRightarrow[R_3 \longrightarrow -2R_1+R_3]{R_2 \longrightarrow -3R_1+R_2} \begin{pmatrix} 1 & -2 & 3 & 3 \\ 0 & 8 & -10 & -3 \\ 0 & 5 & -5 & -2 \end{pmatrix}$$

$$\Rightarrow R_3 \longrightarrow -\frac{5}{8}R_2 + R_3 \begin{pmatrix} 1 & -2 & 3 & 3 \\ 0 & 8 & -10 & -3 \\ 0 & 0 & \frac{5}{4} & -\frac{1}{8} \end{pmatrix}$$ Then back substitution $\begin{matrix} x-2y+3z=3 \\ 8y-10z=-3 \\ \frac{5}{4}z=\frac{-1}{8} \end{matrix}$

$$\Rightarrow z = -\frac{1}{8}.\frac{4}{5} = \frac{-1}{10},$$

$$\Rightarrow\ y = \frac{-3+10z}{8} = \frac{-3+10\left(-\frac{1}{10}\right)}{8} = -\frac{1}{2}\ \ and\ x = 3+2y-3z = 3-1+\frac{3}{10} = \frac{23}{10}$$

solution $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{23}{10} \\ \frac{-1}{2} \\ \frac{-1}{10} \end{pmatrix}$

Note: For a linear system

- $A|B = \begin{pmatrix} 1 & -2 & 3 & 3 \\ 0 & 8 & -10 & -3 \\ 0 & 0 & \frac{5}{4} & -\frac{1}{8} \end{pmatrix}$ has three non-zero rows.
-
- A = $\begin{pmatrix} 1 & -2 & 3 \\ 0 & 8 & -10 \\ 0 & 0 & \frac{5}{4} \end{pmatrix}$ has three non-zero rows
-
- The number of rows of coefficient matrix A is equal to number of rows of augmented matrix A|B
- The system has unique (one) solution.

2. Many(infinite) solution

- The last row of reduced augmented matrix is zero.
- Graph of equations coincide to each other.

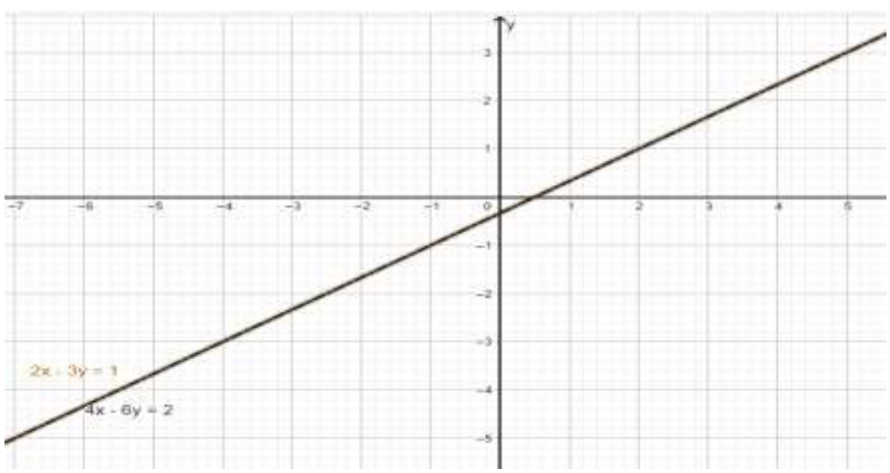
Example

a. Solve $\begin{matrix} 2x - 3y = 1 \\ ,4x - 6y = 2 \end{matrix}$ graphically

Solution

Determine the intersection point and draw the graph

$,\begin{cases} 2x - 3y = 1 \\ 4x - 6y = 2 \end{cases} \Longrightarrow \begin{matrix} -4x + 6y = -2 \\ 4x - 6y = 2 \end{matrix} \Longrightarrow 0 = 0$ the two lines meet at infinitely many points. Solution = $\mathbb{R}$



b. Solve the linear system $\begin{matrix} x \; + \; y - 3z = 3 \\ x - 3y \; + \; 2z = 5 \\ 2x - 2y - z = 8 \end{matrix}$ using Gauss method

Solution

Let A= $\begin{pmatrix}1 & 1 & -3\\ 1 & -3 & 2\\ 2 & -2 & -1\end{pmatrix}$ is the coefficient matrix and A|B= $\begin{pmatrix}1 & 1 & -3 & 3\\ 1 & -3 & 2 & 5\\ 2 & -2 & -1 & 8\end{pmatrix}$ be augmented matrix

$$,\Longrightarrow \frac{R_2 \longrightarrow -R_1 + R_2}{R_3 \longrightarrow -2R_1 + R_3} \begin{pmatrix}1 & 1 & -3 & 3\\ 0 & -4 & 5 & 2\\ 0 & -4 & 5 & 2\end{pmatrix} \Longrightarrow R_3 \longrightarrow -R_2 + R_3 \begin{pmatrix}1 & 1 & -3 & 3\\ 0 & -4 & 5 & 2\\ 0 & 0 & 0 & 0\end{pmatrix}$$

Here you observe that the last row is a zero row. So the system has many solutions.

, back substitution

$$\begin{matrix} x + y - 3z = 3 \\ -4y + 5z = 5 \end{matrix} \Rightarrow \text{here the number of equations is less than number of variables}$$

The solution we fix is **dependent** on a certain variable we fix (parameter).

Let $z = \ t \in R, \Longrightarrow -4y + 5z = 5 \Longrightarrow y = -\frac{5-5z}{4} = \frac{5t-5}{4}$

And $from\ x + y - 3z = 3 \Longrightarrow x = 3 - y + 3z \Longrightarrow x = 3 - \frac{5t-5}{4} + 3t$

$,\Longrightarrow x = \frac{7t}{4} + \frac{17}{4}$

- Therefore our solution is $\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}\frac{7t+17}{4}\\ \frac{5t-5}{4}\\ t\end{pmatrix} = \begin{pmatrix}7t & 17\\ 5t + (-5) & \\ t & 0\end{pmatrix} \ldots .\, dependent$

3. **No Solution (Inconsistent)**
   - The system has no set of values satisfying the equations symaultinously.
   - The lines do not intersect at all.

Example:

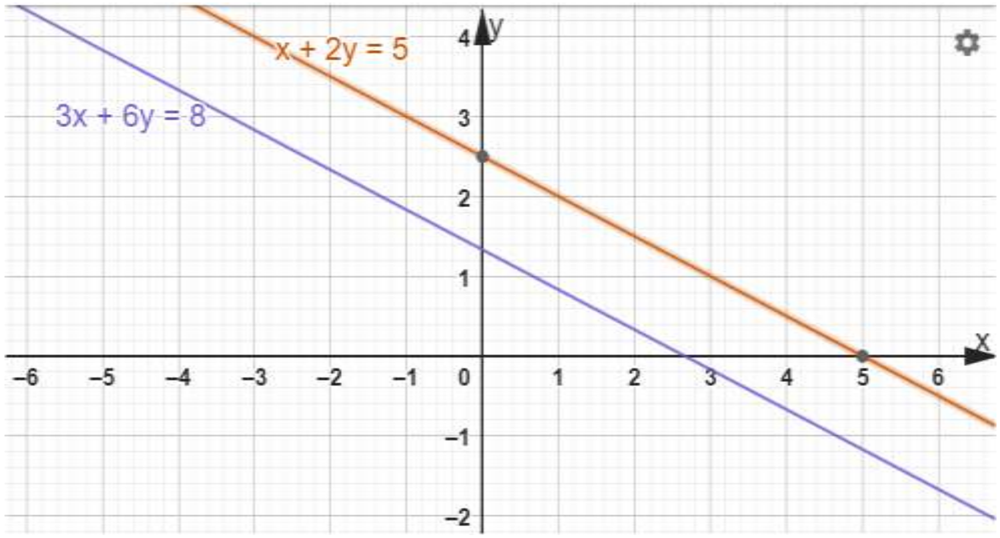
a. Solve $\begin{matrix} x + 2y = 5 \\ 3x + 6y = 8 \end{matrix}$ $graphially$

Solution:

let's find the intersection point of the equations $x = 5 - 2y\ and\ 3(5 - 2y) + 6y = 8$
$\Longrightarrow 15 - 6y + 6y = 8 \Longrightarrow 15 = 8\ is\ false.$

Therefore it has no intersection point.(no solution)

Graphically



b. Solve the system using Gauss method

$$\begin{aligned} x + y - 3z &= 3 \\ x - 3y + 2z &= 5 \\ 2x - 2y - z &= 6 \end{aligned}$$

Solution:

Let A= $\begin{pmatrix} 1 & 1 & -3 \\ 1 & -3 & 2 \\ 2 & -2 & -1 \end{pmatrix}$ = coefficient matrix, A|B= $\begin{pmatrix} 1 & 1 & -3 & 3 \\ 1 & -3 & 2 & 5 \\ 2 & -2 & -1 & 6 \end{pmatrix}$ be augmented matrix

$$, Then \ \Longrightarrow_{R_3 \to -2R_1 + R_3}^{R_2 \to -R_1 + R_2} \begin{pmatrix} 1 & 1 & -3 & 3 \\ 0 & -4 & 5 & 2 \\ 0 & -4 & 5 & 0 \end{pmatrix} \Longrightarrow R_3 \longrightarrow -R_2 + R_3 \begin{pmatrix} 1 & 1 & -3 & 3 \\ 0 & -4 & 5 & 2 \\ 0 & 0 & 0 & -2 \end{pmatrix}$$

The row reduce form shows that the system has no solution.

Note

- A|B = $\begin{pmatrix} 1 & 1 & -3 & 3 \\ 0 & -4 & 5 & 2 \\ 0 & 0 & 0 & -2 \end{pmatrix}$ has three non-zero rows
- $A = \begin{pmatrix} 1 & 1 & -3 \\ 0 & -4 & 5 \\ 0 & 0 & 0 \end{pmatrix}$ has two non-zero rows
- $the\ number\ of\ non-zero\ rows\ of\ A\ is\ less\ than\ the\ number\ non-zero\ rows\ of\ A|B$
- The system has no solution

Examples:

1. Find the values of c for which the system give below has an infinite number of solutions

$$\begin{cases} 2x - 4y = 6 \\ -3x + 6y = c \end{cases}$$

2. For what values of k does $\begin{cases} x+2y-3z=5 \\ 2x-y-z=8 \\ kx+y+2z=14 \end{cases}$ has a unique solution?
3. Find the values of c and d for which both the given points lie on the given straight line $,l: cx+dy=2\,;(1,4) and\ (2\,,16)$.
4. Find a quadratic function $y=ax^2+bx+c$ contains the points $(1\,,9\,)\,,(4\,,6\,) and\ (6\,,14)$?

Solution

1. Let $A|B = \begin{pmatrix} 2 & -4 & 6 \\ -3 & 6 & c \end{pmatrix} \Longrightarrow R_2 \longrightarrow \frac{3}{2}R_1 + R_2 \begin{pmatrix} 2 & -4 & 6 \\ 0 & 0 & c-9 \end{pmatrix}$
   Since the system has many solution, the last row must be zero $\Longrightarrow c-9=0 \Longrightarrow c=9$
2. $\begin{cases} x+2y-3z=5 \\ 2x-y-z=8 \\ kx+y+2z=14 \end{cases}$ $and\ A|B = \begin{pmatrix} 1 & 2 & -3 & 5 \\ 2 & -1 & -1 & 8 \\ k & 1 & 2 & 14 \end{pmatrix}$
   $, \Longrightarrow \frac{R_2 \longrightarrow -2R_1+R_2}{R_3 \longrightarrow -kR_1+R_3} \begin{pmatrix} 1 & 2 & -3 & 5 \\ 0 & -5 & 5 & -2 \\ 0 & 1-2k & 2+3k & 14-5k \end{pmatrix}$
   $\Rightarrow R_2 \longrightarrow -\frac{1}{5}R_2 \begin{pmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & -1 & \frac{2}{5} \\ 0 & 1-2k & 2+3k & 14-5k \end{pmatrix}$
   $, \Longrightarrow R_3 \longrightarrow (2k-1)R_2 + R_3 \begin{pmatrix} 1 & 2 & -3 & 5 \\ 0 & 1 & -1 & \frac{2}{5} \\ 0 & 0 & k+3 & -\frac{24k+68}{5} \end{pmatrix}$
   $,if\ it\ has\ unique\ solution\ k+3 \neq 0 \ \Longrightarrow k \neq -3$

   $, \Longrightarrow \ therefore\ \boldsymbol{k} \in \mathbb{R}\backslash\{-\mathbf{3}\}$

3. $substitute\ \ x\ and\ y\ on\ to\ l: cx+dy=2$
   ,at $(1\,,4) \Longrightarrow c+4d=2\ and\ at\ (2{,}16) \Longrightarrow 2c+16d=2$
   $Then\ it\ gives\ \ a\ system\ \begin{cases} c\ +\ 4d=2 \\ 2c+16d=2 \end{cases}$
   $, \Longrightarrow A|B = \begin{pmatrix} 1 & 4 & 2 \\ 2 & 16 & 2 \end{pmatrix} \Longrightarrow R_2 \longrightarrow -2R_1 + R_2 \begin{pmatrix} 1 & 4 & 2 \\ 0 & 8 & -2 \end{pmatrix}$
   Back substitution $\begin{matrix} c+4d=2 \\ 8d=-2 \end{matrix} \Longrightarrow 8d=-2 \Longrightarrow d=-\frac{1}{4} \ \ and\ c=2-4d=2+1=3$
   Therefore $\begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ -\frac{1}{4} \end{pmatrix}$
4. $y=ax^2+bx+c$ contains the points $(1\,,9\,)\,,(4\,,6\,) and\ (6\,,14)$
   $substitution$
   $at\ \ (1{,}9) \ \Rightarrow 9=a(1)+b(1)+c \ \Longrightarrow a+b+c=9$
   At $(4,6) \Longrightarrow 6=a(4)^2+b(4)+c \Longrightarrow 16a+4b+c=6$
   At $(6{,}14) \Longrightarrow 14=a(6)^2+b(6)+c \Longrightarrow 36a+6b+c=14$

Then the linear system
$$\begin{aligned} a + b + c &= 9 \\ 16a + 4b + c &= 6 \\ 36a + 6b + c &= 14 \end{aligned}$$

Matrix form $AX + B \Longrightarrow \begin{pmatrix} 1 & 1 & 1 \\ 16 & 4 & 1 \\ 36 & 6 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 9 \\ 6 \\ 14 \end{pmatrix}$

Then we solve it by using Gauss method:

$, \Longrightarrow A|B = \begin{pmatrix} 1 & 1 & 1 & 9 \\ 16 & 4 & 1 & 6 \\ 36 & 6 & 1 & 14 \end{pmatrix} \Longrightarrow \frac{R_2 \longrightarrow -16R_1 + R_2}{R_3 \longrightarrow -36R_1 + R_3} \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & -12 & -15 & -138 \\ 0 & -30 & -35 & -310 \end{pmatrix}$

$, \Longrightarrow R_3 \longrightarrow -\frac{30}{12}R_2 + R_3 \begin{pmatrix} 1 & 1 & 1 & 9 \\ 0 & -12 & -15 & -138 \\ 0 & 0 & \frac{5}{2} & 35 \end{pmatrix}$

Back substitution
$$\begin{aligned} a + b + c &= 9 \\ -12b - 15c &= -138 \\ \frac{5}{2}c &= 35 \end{aligned}$$

$, \Longrightarrow \frac{5}{2}c = 35 \Longrightarrow c = 14, \ -12b - 15c = -138 \Longrightarrow b = -\frac{15c - 138}{12} = -\frac{210 - 138}{12} = -6$

$, \Longrightarrow a + b + c = 9 \Longrightarrow a = 9 - b - c \Longrightarrow a = 9 + 6 - 14 = 1$

Therefore the function $y = ax^2 + bx + c \ \Longrightarrow y = x^2 - 6x + 14$