Lesson 4: Inverse of Square Matrices

Definition:
A square matrix A is said to be invertible or non- singular if and only if there is a square matrix B such that $A\,B \;=\; BA \;=\; I$, where is an identity matrix having the same order as A.

For a square matrix A

- The inverse of A is denoted by A^{-1}. And $A^{-1} \neq O$
- A matrix that doesn't have an inverse is singular

Example :

1. Find the inverse of a matrix given below

   a. $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ b $.B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 1 & 4 \end{pmatrix}$

Solution

a. If A = $\begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}$ is invertible then there is a matrix B = $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, such that AB =I =BA where I = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$:

$,\Longrightarrow \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Longrightarrow \begin{pmatrix} 3a + c & 3b + d \\ a + 2c & b + 2d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$,\Longrightarrow 3a + c = 1 \; and \; a + 2c = 0 \Longrightarrow 3(-2c) + c = 1 \Longrightarrow -5c = 1 \Longrightarrow c = -\frac{1}{5},$

$,a = -2\left(-\frac{1}{5}\right) = \frac{2}{5}$

$,3b + d = 0 \; and \; b + 2d = 1 \Longrightarrow b + 2(-3b) = 1 \Longrightarrow b = -\frac{1}{5}, d = \frac{3}{5}$

Therefore $A^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix}$

b. If the matrix is a 3 x 3 or more matrix we use **Gauss- Jordan method** (row reduced echelon form) to find it inverse

$, B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 1 \\ 2 & 1 & 4 \end{pmatrix}$ and I = $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Steps to do

Step1 – take A|I as an augmented matrix

Step 2- reduce A till A is to be changed in to identity matrix

$$,\Rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 1 & 0 & 1 & 0 \\ 2 & 1 & 4 & 0 & 0 & 1 \end{array}\right) \Rightarrow \frac{R_2 \longrightarrow -4R_1 + R_2}{R_3 \longrightarrow -2R_1 + R_3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -11 & -4 & 1 & 0 \\ 0 & -3 & -2 & -2 & 0 & 1 \end{array}\right)$$

$$,\Longrightarrow R_2 \longrightarrow -\frac{R_2}{3} \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & \frac{11}{3} & \frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & -3 & -2 & -2 & 0 & 1 \end{array}\right)$$

$$,\xrightarrow[R_3\to 3R_2+R_3]{R_1\to -2R_2+R_1}\left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{13}{3} & -\frac{5}{3} & \frac{2}{3} & 0 \\ 0 & 1 & \frac{11}{3} & \frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 9 & 2 & -1 & 1 \end{array}\right)$$

$$\Rightarrow R_3 \longrightarrow \frac{1}{9}R_3 \left(\begin{array}{ccc|ccc} 1 & 0 & -\frac{13}{3} & -\frac{5}{3} & \frac{2}{3} & 0 \\ 0 & 1 & \frac{11}{3} & \frac{4}{3} & -\frac{1}{3} & 0 \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{1}{9} & \frac{1}{9} \end{array}\right)$$

$$\xrightarrow[R_2\to -\frac{11}{3}R_3+R_2]{R_1\to \frac{13}{3}R_3+R_1}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -\frac{19}{27} & \frac{2}{9} & \frac{13}{27} \\ 0 & 1 & 0 & \frac{14}{27} & \frac{2}{27} & -\frac{11}{27} \\ 0 & 0 & 1 & \frac{2}{9} & -\frac{1}{9} & \frac{1}{9} \end{array}\right)$$,is the last step

$$therefore\ B^{-1} = \begin{pmatrix} -\frac{19}{27} & \frac{2}{9} & \frac{13}{27} \\ \frac{14}{27} & \frac{2}{27} & -\frac{11}{27} \\ \frac{2}{9} & -\frac{1}{9} & \frac{1}{9} \end{pmatrix}$$

Solving linear system of equations using $\boldsymbol{AX = B \Longrightarrow X = A^{-1}B}$

Example

1. Solve $\begin{matrix} 3x + y = 5 \\ x + 2y = -3 \end{matrix}$

Solution

, First describe the system in matrix form $A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}, X = \begin{pmatrix} x \\ y \end{pmatrix}, B = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$

Then $AX = B \Longrightarrow X = A^{-1}B$

,step1:- find $A^{-1} \Longrightarrow A|I = \left(\begin{array}{cc|cc} 3 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \end{array}\right)$

, $\Rightarrow R_1 \leftrightarrow R_2 \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 3 & 1 & 1 & 0 \end{array}\right) \Longrightarrow R_2 \leftrightarrow -3R_1 + R_2 \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & -5 & 1 & -3 \end{array}\right)$

$R_2 \longrightarrow -\frac{1}{5}R_2 \left(\begin{array}{cc|cc} 1 & 2 & 0 & 1 \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \end{array}\right) \Rightarrow R_1 \longrightarrow -2R_2 + R_1 \left(\begin{array}{cc|cc} 1 & 0 & \frac{2}{5} & -\frac{1}{5} \\ 0 & 1 & -\frac{1}{5} & \frac{3}{5} \end{array}\right)$

$therefore\ A^{-1} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix}$

$step2: -find\ X\ by, \quad X = A^{-1}B$

$$\Rightarrow \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{5} & \frac{3}{5} \end{pmatrix} \begin{pmatrix} 5 \\ -3 \end{pmatrix} = \begin{pmatrix} \frac{13}{5} \\ -\frac{14}{5} \end{pmatrix}$$

Properties of Invertible Matrices

Let A and B be invertible matrices of the same size. Then

- $A^{-1} is\ invertible\ and\ (A^{-1})^{-1} = A$
- $A^{t} is\ invertible\ and\ (A^{t})^{-1} = (\mathrm{A}^{-1})^{\mathrm{t}}$
- $AB\ is\ invertible\ and\ (AB)^{-1} = \mathrm{B}^{-1}\mathrm{A}^{-1}$

Examples

Let A= $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$ and B= $\begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix}$ then find

a. A^{-1} b. $(A^{-1})^{-1}$ c. B^{-1} d. $(AB)^{-1}\ and\ B^{-1}A^{-1}$ e. $(A^{t})^{-1} and\ (A^{-1})^{t}$

Solution:

a. $A^{-1} \Rightarrow \left(\begin{matrix} 2 & 3 \\ 1 & 2 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right) \Longrightarrow R_1 \leftrightarrow R_2 \left(\begin{matrix} 1 & 2 \\ 2 & 3 \end{matrix} \middle| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}\right)$

$\Rightarrow R_2 \longrightarrow -2R_1 + R_2 \left(\begin{matrix} 1 & 2 \\ 0 & -1 \end{matrix} \middle| \begin{matrix} 0 & 1 \\ 1 & -2 \end{matrix}\right) \Rightarrow R_2 \leftrightarrow -1R_2 \left(\begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 0 & 1 \\ -1 & 2 \end{matrix}\right)$

$, \Rightarrow R_1 \longrightarrow -2R_2 + R_1 \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 2 & -3 \\ -1 & 2 \end{matrix}\right)$

$\therefore A^{-1} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

$b. (A^{-1})^{-1} \Rightarrow \left(\begin{matrix} 2 & -3 \\ -1 & 2 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right) \Rightarrow R_1 \longrightarrow \frac{1}{2}R_1 \left(\begin{matrix} 1 & -\frac{3}{2} \\ -1 & 2 \end{matrix} \middle| \begin{matrix} \frac{1}{2} & 0 \\ 0 & 1 \end{matrix}\right)$

$, \Longrightarrow R_2 \longrightarrow R_1 + R_2 \left(\begin{matrix} 1 & -\frac{3}{2} \\ 0 & \frac{1}{2} \end{matrix} \middle| \begin{matrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 1 \end{matrix}\right) \Longrightarrow R_1 \longrightarrow 3R_2 + R_1 \left(\begin{matrix} 1 & 0 \\ 0 & \frac{1}{2} \end{matrix} \middle| \begin{matrix} 2 & 3 \\ \frac{1}{2} & 1 \end{matrix}\right)$

$\Longrightarrow R_2 \longrightarrow 2R_2 \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \middle| \begin{matrix} 2 & 3 \\ 1 & 2 \end{matrix}\right)$

$therefoe (A^{-1})^{-1} = A = \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$

C .B= $\begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} \Longrightarrow \mathrm{B}^{-1} \Longrightarrow \left(\begin{matrix} 0 & 2 \\ 3 & 1 \end{matrix} \middle| \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right) \Rightarrow R_1 \leftrightarrow R_2 \left(\begin{matrix} 3 & 1 \\ 0 & 2 \end{matrix} \middle| \begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}\right)$

$$\Rightarrow R_1 \leftrightarrow R_2 \left(\begin{array}{cc|cc} 3 & 1 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{array}\right) \Longrightarrow \frac{R_1 \longrightarrow \frac{1}{3}R_1}{R_2 \longrightarrow \frac{1}{2}R_2} \left(\begin{array}{cc|cc} 1 & \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right)$$

$$\Longrightarrow R_1 \longrightarrow -\frac{1}{3}R_2 + R_1 \left(\begin{array}{cc|cc} 1 & 0 & -\frac{1}{6} & \frac{1}{3} \\ 0 & 1 & \frac{1}{2} & 0 \end{array}\right)$$

$$\therefore B^{-1} = \begin{pmatrix} -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & 0 \end{pmatrix}$$

$d..\ (AB)^{-1}\ and\ B^{-1}A^{-1}$

.AB = $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 0+9 & 4+3 \\ 0+6 & 2+2 \end{pmatrix} = \begin{pmatrix} 9 & 7 \\ 6 & 4 \end{pmatrix}$

And $(AB)^{-1} \Longrightarrow \left(\begin{array}{cc|cc} 9 & 7 & 1 & 0 \\ 6 & 4 & 0 & 1 \end{array}\right) \Longrightarrow R_2 \longrightarrow -\frac{6}{9}R_1 + R_2 \left(\begin{array}{cc|cc} 9 & 7 & 1 & 0 \\ 0 & -\frac{2}{3} & -\frac{2}{3} & 1 \end{array}\right)$

$$\Longrightarrow \frac{R_1 \longrightarrow \frac{1}{9}R_1}{R_2 \longrightarrow -\frac{3}{2}R_2} \left(\begin{array}{cc|cc} 1 & \frac{7}{9} & \frac{1}{9} & 0 \\ 0 & 1 & 1 & -\frac{3}{2} \end{array}\right) \Longrightarrow R_1 \longrightarrow -\frac{7}{9}R_2 + R_1 \left(\begin{array}{cc|cc} 1 & 0 & \frac{-2}{3} & \frac{7}{6} \\ 0 & 1 & 1 & -\frac{3}{2} \end{array}\right)$$

$$\therefore (AB)^{-1} = \begin{pmatrix} \frac{-2}{3} & \frac{7}{6} \\ 1 & -\frac{3}{2} \end{pmatrix}$$

$then\ we\ need\ to\ show\ that\ \ (AB)^{-1} = B^{-1}A^{-1}$

$$B^{-1}A^{-1} = \begin{pmatrix} -\frac{1}{6} & \frac{1}{3} \\ \frac{1}{2} & 0 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}-\frac{1}{3} & \frac{1}{2}+\frac{2}{3} \\ 1+0 & -\frac{3}{2}+0 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} & \frac{7}{6} \\ 1 & -\frac{3}{2} \end{pmatrix}$$

$$\therefore (AB)^{-1} = \begin{pmatrix} \frac{-2}{3} & \frac{7}{6} \\ 1 & -\frac{3}{2} \end{pmatrix} = B^{-1}A^{-1}$$

e. $(A^t)^{-1} and\ (A^{-1})^t$

let's find . $(A^t)^{-1}$ first $A^t = \begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix} \Rightarrow (A^t)^{-1}|I = \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 3 & 2 & 0 & 1 \end{array}\right)$

$$\Longrightarrow R_2 \longrightarrow -\frac{3}{2}R_1 + R_2 \left(\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & \frac{1}{2} & -\frac{3}{2} & 1 \end{array}\right) \Rightarrow \frac{R_1 \longrightarrow \frac{1}{2}R_1}{R_2 \longrightarrow 2R_2} \left(\begin{array}{cc|cc} 1 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -3 & 2 \end{array}\right)$$

$$\Rightarrow R_1 \longrightarrow -\frac{1}{2}R_2 + R_1 \left(\begin{array}{cc|cc} 1 & 0 & 2 & -1 \\ 0 & 1 & -3 & 2 \end{array}\right)$$

$\therefore (A^t)^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$ and $(A^{-1})^t = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}^t = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix}$

$$\therefore (A^t)^{-1} = \begin{pmatrix} 2 & -1 \\ -3 & 2 \end{pmatrix} = (A^{-1})^t$$

2. Use elementary row operation to a given matrix is invertible or not

a. $A = \begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ b. $B = \begin{pmatrix} 4 & -2 \\ -2 & 1 \end{pmatrix}$

Solution:

a. Let A be invertible, then $A^{-1} is\ a\ non-zero\ matrix,$ determined using elementary row operation we did before

Then $A|I = \left(\begin{matrix} 1 & 2 \\ 1 & 3 \end{matrix}\middle|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right)$ reduce rows until A is changed in to an identity matrix

$\Rightarrow R_2 \longrightarrow -R_1 + R_2 \left(\begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} 1 & 0 \\ -1 & 1 \end{matrix}\right)$

$\Rightarrow R_1 \longrightarrow -2R_2 + R_1 \left(\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\middle|\begin{matrix} 3 & -2 \\ -1 & 1 \end{matrix}\right)$, a is transformed in to identity

$\Rightarrow A^{-1} = \begin{pmatrix} 3 & -2 \\ -1 & 1 \end{pmatrix}$

b. Let's assume B is invertible then B|I= $\left(\begin{matrix} 4 & -2 \\ -2 & 1 \end{matrix}\middle|\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\right)$

$\Rightarrow R_2 \longrightarrow \frac{1}{2}R_1 + R_2 \left(\begin{matrix} 4 & -2 \\ 0 & 0 \end{matrix}\middle|\begin{matrix} 1 & 0 \\ \frac{1}{2} & 1 \end{matrix}\right)$....... It shows;

- B cannot be changed in to an identity matrix
- B is not invertible.

Lesson 7: Application Problems

Example

1. In a triangle the smallest angle measures 20^0 more than one – third of the largest angle. The middle angle measures 15^0 $more\ than\ the\ smallest\ angle$. Find the measure of each angle of the triangle?
2. The perimeter of a given triangle is 32 cm. the shortest side is 6cm shorter than the largest side of the given triangle, the longest side is 10cm less than the sum of the other two sides of a triangle. Find the measure of lengths of sides of a triangle?
3. A chemist has 200 milliliter of 80% of methane solution. How much of a 50 % solution must she added to the final solution is 60% methane.
4. A train travels 70 km in the same time that a truck travel 50km. find the speed of each vehicles if the train's average speed is 10 km/h faster than the truck's speed?
5. The farmer has two types of milk, one that is 20% butter fat and another which is 50% butter fat . how much of each should he use end up with 50 liter of 20% butter fat?

Solution:

1. Let x be the measure of smallest angle of a triangle
   - ✓ y be the measure of the middle
   - ✓ Z be the measure of largest angle

From the above information

$x + y + z = 180$

- $x = \frac{1}{2}z + 20$

$y = 15 + x$

- The system of linear equation is of the form $\begin{matrix} x + y + z = 180 \\ 2x - z = 20 \\ y - x = 15 \end{matrix}$
- Matrix form of the equation $A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ $and\ B = \begin{pmatrix} 180 \\ 20 \\ 15 \end{pmatrix}$
- Augmented matrix becomes A|B= $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 180 \\ 2 & 0 & -1 & 20 \\ -1 & 1 & 0 & 15 \end{array}\right)$
- Row reductions leads to $\Longrightarrow \frac{R_2 \longrightarrow -2R_1 + R_2}{R_3 \longrightarrow R_1 + R_3} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 180 \\ 0 & -2 & -3 & -340 \\ 0 & 2 & 1 & 195 \end{array}\right)$

$\Longrightarrow R_3 \longrightarrow R_2 + R_3 \left(\begin{array}{ccc|c} 1 & 1 & 1 & 180 \\ 0 & -2 & -3 & -340 \\ 0 & 0 & -2 & -145 \end{array}\right)$

Back substitution

$\mathbf{2z = 145 \Longrightarrow z = 72.5}$

$\mathbf{-2y - 3z = -340 \ \Longrightarrow y = \frac{-340 + 3(72.5)}{-2} = \frac{122.5}{2} = 61.5}$

$\mathbf{x + y + z = 180 \Longrightarrow x = 180 - (72.5 + 61.5) = 46}$

Answer

- the smallest angle measures 46^0
- the middle angle measures 61.25^0
- the largest angle measures 72.5^0

2. let p be perimeter of a triangle
   - x be the length of shortest side
   - y be the length of middle side
   - z length of longest side

Then the longest side is 10cm less than the sum of the other two sides $\Rightarrow z = x + y - 10$

the shortest side is 6cm shorter than the largest side , $\Rightarrow$ x = z − 6

Perimeter 32cm $= x + y + z$

The linear system $\begin{matrix} x + y + z = 32 \\ z = x + y - 10 \\ x = z - 6 \end{matrix} \Longrightarrow \begin{cases} x + y + z = 32 \\ x + y - z = 10 \\ -x + z = 6 \end{cases}$

Augmented matrix $\begin{pmatrix} 1 & 1 & 1 & 32 \\ 1 & 1 & -1 & 10 \\ -1 & 1 & 0 & 6 \end{pmatrix} \Rightarrow \frac{R_2 \longrightarrow -R_1 + R_2}{R_3 \longrightarrow R_1 + R_3} \begin{pmatrix} 1 & 1 & 1 & 32 \\ 0 & 0 & -2 & -22 \\ 0 & 2 & 1 & 38 \end{pmatrix}$

$\Rightarrow R_2 \leftrightarrow R_3 \begin{pmatrix} 1 & 1 & 1 & 32 \\ 0 & 2 & 1 & 38 \\ 0 & 0 & -2 & -22 \end{pmatrix}$

Back substitution: $\begin{aligned} x + y + z &= 32 \\ 2y + z &= 38 \\ -2z &= -22 \end{aligned}$

,$\Longrightarrow z = 11, 2y + z = 38 \Longrightarrow y = \frac{38-z}{2} = 13.5 \; and \; x = 32 - y - z = 32 - 11 - 13.5 = 7.5$

Answer: the shortest side measures 7.5 cm
The middle side measures 11 cm
And the longest side measures 13.5 cm