Lesson 2: **Introduction to matrices**

Definition

Let A be square matrix with order n. then determinant of A is a scalar value of a function of entries of A. The determinant of A is denoted by:

- $det(A)\ or$
- $|A|$
- $|A| \in R$

Example

1. Let $A = \begin{pmatrix} a & b & c \\ e & f & g \\ p & q & r \end{pmatrix}_{3\times 3}$ then

   a. Determinant of A= $\det(A) = \begin{vmatrix} a & b & c \\ e & f & g \\ p & q & r \end{vmatrix}$

4.2. **Determinants of Matrices with Order 2**

> Definition:
>
> Let A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $be\ a\ 2 \times 2\ matrix. then\ determinant\ of\ A\ is:$
>
> - det(A)= $|A| = ad - bc$

Example

1.$let\ A = \begin{pmatrix} 4 & 5 \\ 7 & 2 \end{pmatrix}$, then find $a. det(A)$ $b. det(2A)$

2. Let A= $\begin{pmatrix} x & 3+x \\ 1 & 2x \end{pmatrix}$ and $|A| = 3$, Then find x

3. Let A= $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det(A) = 4$ $then\ find\ determinant\ of\ the\ following\ matrices$

   a. $\begin{pmatrix} a & b \\ 3a+c & 3b+d \end{pmatrix}$ b. $\begin{pmatrix} 5a & 5b \\ 5c & 5d \end{pmatrix}$ c. $\begin{pmatrix} 4a & 4b \\ c & d \end{pmatrix}$ d. $\begin{pmatrix} \frac{d}{4} & -\frac{b}{4} \\ -\frac{c}{4} & \frac{a}{4} \end{pmatrix}$

4. Determine determinant of a matrix I = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Solution

1. $A = \begin{pmatrix} 4 & 5 \\ 7 & 2 \end{pmatrix}$ a. $det(A) = \begin{vmatrix} 4 & 5 \\ 7 & 2 \end{vmatrix} = 4 \times 2 - 5 \times 7 = 8 - 35 = -27$

   b. $det(2A) = 2\begin{pmatrix} 4 & 5 \\ 7 & 2 \end{pmatrix} = \begin{pmatrix} 2\times 4 & 2\times 5 \\ 2\times 7 & 2\times 2 \end{pmatrix} = \begin{vmatrix} 8 & 10 \\ 14 & 4 \end{vmatrix} = 8 \times 4 - 10 \times 14 = -108$

2. A= $\begin{pmatrix} x & 3+x \\ 1 & 2x \end{pmatrix}$ and $|A| = 3$

   $\det(A) = \begin{vmatrix} x & 3+x \\ 1 & 2x \end{vmatrix} = 3$

   $\Longrightarrow 2x^2 - (3+x) = 3 \Longrightarrow 2x^2 - x - 3 = 3$

   $\Longrightarrow 2x^2 - x - 6 = 0$

   $\Longleftrightarrow 2x^2 - 4x + 3x - 6 = 0$

   $\Longleftrightarrow 2x(x-2) + 3(x-2) = 0$

$\Leftrightarrow (2x+3)(x-2)=0 \Longrightarrow 2x+3=0 \; or \; x-2=0$

$\Longrightarrow x=-\frac{3}{2} \, or \; x=2$

3. $A=\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\det(A)=4$

$\det(A)=4 \Longrightarrow \begin{vmatrix} a & b \\ c & d \end{vmatrix}=4 \Longrightarrow ad-bc=4$

a. $\begin{pmatrix} a & b \\ 3a+c & 3b+d \end{pmatrix} \Longrightarrow \begin{vmatrix} a & b \\ 3a+c & 3b+d \end{vmatrix}=a(3b+d)-b(3a+c)$

$=3ab+ad-3ab-bc$

$=3ab-3ab+ad-bc=ad-bc=4$

b. $\begin{pmatrix} 5a & 5b \\ 5c & 5d \end{pmatrix}=\begin{vmatrix} 5a & 5b \\ 5c & 5d \end{vmatrix}=5a\times 5d-5b\times 5c=25ad-25bc$

$=25(ad-bc)=25\times 4=100$

c. $\begin{pmatrix} 4a & 4b \\ c & d \end{pmatrix}=4ad-4bc=4(ad-bc)=4\times 4=16$

d. $\begin{pmatrix} \frac{d}{4} & -\frac{b}{4} \\ -\frac{c}{4} & \frac{a}{4} \end{pmatrix}=\frac{ad}{16}-\frac{bc}{16}=\frac{1}{16}(ad-bc)=\frac{4}{16}=\frac{1}{4}$

4. $I=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}=1.1-0.0=1-0=1$

4.3. Minors And Cofactors of Elements of Matrices

Definition:

Let $A=(a_{ij})_{n\times n}$ be a matrix and M be the sub-matrix of A obtained by deleting the $i^{th}-raw\ and\ j^{th}$ −Column of A for $i;\ j = 1,2,3,\ldots n$. Then:

I. The minor for A at location $(i;\ j)$, denoted by $M_{ij}(A)$, is the determinant of the sub matrix A_{ij}.

$\boldsymbol{M_{ij}(A) = det(A_{ij})}$.

II. The cofactor, denoted by $\Delta_{ij}(A)$, for A at location (i,j) is the signed determinant of the sub matrix A_{ij}.

$\Delta\,\boldsymbol{_{ij}(A)=(-1)^{i+j}\,M_{ij}(A)}$.

Remark.

- the cofactor $C_{ij}(A)$ at location (i,j) can be computed as

$$C_{ij}(A)=\begin{cases} \det(A_{ij}), \text{if } i+j \text{ is even} \\ -\det(A_{ij}), \text{ if } i+j \text{ is odd} \end{cases}$$

Example:

1. let $A=\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$, then find all minors and cofactors of A

Solution:

All the minors of A are:

$M_{11} = deleate\ first\ row\ and\ first\ column = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{23}a_{32}$

$M_{12} = deleate\ first\ row\ and\ second\ column = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{33}$

$M_{13} = deleate\ first\ row\ and\ third\ ccolumn = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} = a_{21}a_{32} - a_{22}a_{31}$

$M_{21} = deleate\ second\ row\ and\ first\ column = \begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix} = a_{12}a_{33} - a_{13}a_{32}$

$M_{22} = deleate\ second\ row\ and\ second\ column = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{23}a_{31}$

$M_{23} = deleate\ second\ row\ and\ third\ column = \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} = a_{11}a_{32} - a_{12}a_{31}$

$M_{31} = deleate\ third\ row\ and\ first\ column = \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix} = a_{12}a_{23} - a_{13}a_{22}$

$M_{32} = deleate\ third\ row\ and\ second\ column = \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} = a_{11}a_{23} - a_{13}a_{21}$

$M_{33} = deleate\ third\ row\ and\ third\ column = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$

$\boldsymbol{and\ all\ the\ cofactors\ of\ A\ are}$:

$\boldsymbol{C}_{\boldsymbol{ij}}(\boldsymbol{A}) = (-\mathbf{1})^{i+j}\ \boldsymbol{M}_{ij}$

$\boldsymbol{C}_{\mathbf{11}} = (-\mathbf{1})^{1+1}\ \boldsymbol{M}_{11} = \boldsymbol{M}_{11} = a_{22}a_{33} - a_{23}a_{32}$

$\boldsymbol{C}_{\mathbf{12}} = (-\mathbf{1})^{1+2}\ \boldsymbol{M}_{12} = -\boldsymbol{M}_{12} = -a_{21}a_{33} + a_{23}a_{33}$

$\boldsymbol{C}_{\mathbf{13}} = (-\mathbf{1})^{1+3}\ \boldsymbol{M}_{13} = \boldsymbol{M}_{13} = a_{21}a_{32} - a_{22}a_{31}$

$\boldsymbol{C}_{\mathbf{21}} = (-\mathbf{1})^{2+1}\ \boldsymbol{M}_{21} = -\boldsymbol{M}_{21} = -a_{12}a_{33} + a_{13}a_{32}$

$\boldsymbol{C}_{\mathbf{22}} = (-\mathbf{1})^{2+2}\ \boldsymbol{M}_{22} = \boldsymbol{M}_{22} = a_{21}a_{33} - a_{23}a_{31}$

$\boldsymbol{C}_{\mathbf{23}} = (-\mathbf{1})^{2+3}\ \boldsymbol{M}_{23} = \boldsymbol{M}_{23} = -a_{11}a_{32} + a_{12}a_{31}$

$\boldsymbol{C}_{\mathbf{31}} = (-\mathbf{1})^{3+1}\ \boldsymbol{M}_{31} = \boldsymbol{M}_{31} = a_{12}a_{23} - a_{13}a_{22}$

$\boldsymbol{C}_{\mathbf{32}} = (-\mathbf{1})^{3+2}\ \boldsymbol{M}_{32} = \boldsymbol{M}_{32} = -a_{11}a_{23} + a_{13}a_{21}$

$\boldsymbol{C}_{\mathbf{33}} = (-\mathbf{1})^{3+3}\ \boldsymbol{M}_{33} = \boldsymbol{M}_{33} = a_{11}a_{22} - a_{12}a_{21}$

1. Let A= $\begin{pmatrix} 1 & 2 & 3 \\ 5 & 2 & 1 \\ 0 & 3 & 2 \end{pmatrix}$ the find

- $\boldsymbol{a}$. the minors and cofactors of A
- $\boldsymbol{b}$. $\boldsymbol{find}\ \boldsymbol{a}_{11}\Delta_{11} + \boldsymbol{a}_{12}\Delta_{12} + \boldsymbol{a}_{13}\Delta_{13}$

Solution:

Minors of A **cofactors of A**

- $M_{11} = \begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 2 \times 2 - 1 \times 3 = 1,$ $\boldsymbol{C}_{\mathbf{11}} = (-\mathbf{1})^{1+1}\ \boldsymbol{M}_{11} = \mathbf{1}$

- ➨ $M_{12} = \begin{vmatrix} 5 & 1 \\ 0 & 2 \end{vmatrix} = 5 \times 2 - 0 \times 1 = 10.$, $\boldsymbol{C}_{\mathbf{12}} = (-\mathbf{1})^{\mathbf{1+2}}\, \boldsymbol{M}_{\mathbf{11}} = -\mathbf{10}$
- ➨ $M_{13} = \begin{vmatrix} 5 & 2 \\ 0 & 3 \end{vmatrix} = 5 \times 3 - 0 \times 2 = 15,$ $\boldsymbol{C}_{\mathbf{13}} = (-\mathbf{1})^{\mathbf{1+3}}\, \boldsymbol{M}_{\mathbf{13}} = \mathbf{15}$
- ➨ $M_{21} = \begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 2 \times 2 - 3 \times 3 = -5.$ $\boldsymbol{C}_{\mathbf{21}} = (-\mathbf{1})^{\mathbf{2+1}}\, \boldsymbol{M}_{\mathbf{21}} = \mathbf{5}$
- ➨ $M_{22} = \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = 1 \times 2 - 0 \times 3 = 2$, $|\boldsymbol{C}_{\mathbf{22}} = (-\mathbf{1})^{\mathbf{2+2}}\, \boldsymbol{M}_{\mathbf{22}} = \mathbf{2}$
- ➨ $M_{23} = \begin{vmatrix} 1 & 3 \\ 0 & 2 \end{vmatrix} = 1 \times 2 - 0 \times 3 = 2$ $\boldsymbol{C}_{\mathbf{23}} = (-\mathbf{1})^{\mathbf{2+3}}\, \boldsymbol{M}_{\mathbf{23}} = -\mathbf{2}$
- ➨ $M_{31} = \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} = 2 \times 1 - 2 \times 3 = -4,$ $\boldsymbol{C}_{\mathbf{31}} = (-\mathbf{1})^{\mathbf{3+1}}\, \boldsymbol{M}_{\mathbf{31}} = -\mathbf{4}$
- ➨ $M_{32} = \begin{vmatrix} 1 & 3 \\ 5 & 1 \end{vmatrix} = 1 \times 1 - 3 \times 5 = -14,$ $\boldsymbol{C}_{\mathbf{32}} = (-\mathbf{1})^{\mathbf{3+2}}\, \boldsymbol{M}_{\mathbf{32}} = \mathbf{14}$
- ➨ $M_{33} = \begin{vmatrix} 1 & 2 \\ 5 & 2 \end{vmatrix} = 1 \times 2 - 2 \times 5 = -8,$ $\boldsymbol{C}_{\mathbf{33}} = (-\mathbf{1})^{\mathbf{3+3}}\, \boldsymbol{M}_{\mathbf{33}} = -\mathbf{8}$

b. A= $\begin{pmatrix} 1 & 2 & 3 \\ 5 & 2 & 1 \\ 0 & 3 & 2 \end{pmatrix}$, $\Delta_{\mathbf{11}} = 1,$ $\Delta_{\mathbf{12}} = -10,$ $,\Delta_{\mathbf{13}} = 15$

$\boldsymbol{a}_{11}\Delta_{\mathbf{11}} + \boldsymbol{a}_{\mathbf{12}}\Delta_{\mathbf{12}} + \boldsymbol{a}_{\mathbf{13}}\Delta_{\mathbf{13}} = \mathbf{1} \times \mathbf{1} + \mathbf{2} \times (-\mathbf{10}) + \mathbf{3} \times \mathbf{15} = \mathbf{1} - \mathbf{2} + \mathbf{45} = \mathbf{44}$

2. Find minors and cofactors of A= $\begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$ and B= (4)

Solution

a. Minors of A

$M_{11} = |3| = 3$
$M_{12} = |5| = 5$
$M_{21} = |1| = 1$
$M_{22} = |2| = 2$

Cofactors of A

$\Delta_{11} = M_{11} = 3$
$\Delta_{12} = -M_{11} = -5$
$\Delta_{11} = M_{11} = 1$
$\Delta_{11} = M_{11} = \mathbf{2}$

b. B= (4) , the matrix has no minor and cofactor.

> **Note**
> The sign $(-1)^{i+j}$on the cofactor of a matrix is a pattern with +'s on the main diagonal.
> $$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

4.3. **Determinant of A square Matrix with Order 3**

Definition: Let A= $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$ be a 3×3 matrix then

$$\det(A) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Such that $det(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$ ……. *first row expansion*

$= a_{21}C_{21} + a_{22}C_{22} + a_{23}C_{23}$ ………. *second row expansion*

$= a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33}$ ……….. *third row expansion* is generally said to be the cofactor expansion along rows.

Examples:

1. Find the determinant of the following matrices.

a. A= $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ 2 & 3 & 5 \end{pmatrix}$ b. B= $\begin{pmatrix} 2 & 3 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}$ c. D= $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 0 & 0 \end{pmatrix}$ d. C= $\begin{pmatrix} 2 & 1 & 1 \\ 3 & 2 & 0 \\ 2 & 3 & 3 \end{pmatrix}$

Solution:

a. A= $\begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ 2 & 3 & 5 \end{pmatrix}$

$$\Rightarrow \det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 0 \\ 2 & 3 & 5 \end{vmatrix} = 1 \times \begin{vmatrix} 1 & 0 \\ 3 & 5 \end{vmatrix} + (-2)\begin{vmatrix} 3 & 0 \\ 2 & 5 \end{vmatrix} + 3\begin{vmatrix} 3 & 1 \\ 2 & 3 \end{vmatrix}$$

$$= 1(1 \times 5 - 0 \times 3) + (-2)(3 \times 5 - 2 \times 0) + 3(3 \times 3 - 2 \times 1)$$

$$= (5 - 0) + (-2)(15 - 0) + 3(9 - 2)$$

$$= 5 - 30 + 21$$

$$det(A) = -13$$

b. $B = \begin{pmatrix} 2 & 3 & 0 \\ 1 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix} \Rightarrow \det(B) = 2\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} + (-3)\begin{vmatrix} 1 & 2 \\ 0 & 0 \end{vmatrix} + 0\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}$

$$= 2(1 \times 0 - 2 \times 2) + (-3)(1 \times 0 - 0 \times 2) + 0(1 \times 2 - 0 \times 1)$$

$$=2(0 - 4) - 3(0) + 0(2 - 0)$$

$$= -8$$

$$\therefore \det(B) = -8$$