Lesson 2: Properties of Determinants of Matrices

1. let A = $\left(a_{ij}\right)_{n\times n}$ be a diagonal or a triangular matrix then $\boldsymbol{\det(A)}$ ***is the*** **product of its diagonal elements**

   Example:

   Find determinants of A= $\begin{pmatrix} -2 & 2 & 3 & 2 & 2 & 3 \\ 0 & 8 & 4 & 4 & 5 & 6 \\ 0 & 0 & 1 & 4 & 8 & 9 \\ 0 & 0 & 0 & 4 & 8 & 9 \\ 0 & 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{pmatrix}$ and B = $\begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$

   Solution:

   A= $\begin{pmatrix} -2 & 2 & 3 & 2 & 2 & 3 \\ 0 & 8 & 4 & 4 & 5 & 6 \\ 0 & 0 & 1 & 4 & 8 & 9 \\ 0 & 0 & 0 & 4 & 8 & 9 \\ 0 & 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 0 & 9 \end{pmatrix}$ $\mathit{is\ the\ upper\ triangular\ matrix,}$

   $\Longrightarrow \det(A) = -2\times 8\times 1\times 4\times 1\times 9 = -576$

   B = $\begin{pmatrix} 8 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ $\mathit{is\ a\ diagonal\ matrix, then}\ \det(B) = 8\times 2\times 4\times 3 = 192$

2. Interchanging rows or columns of a square matrix changes only the sign of its determinant.

   i.e. if $A \xrightarrow{R_i \leftrightarrow R_j} B\ for\ i \neq j$, then $|A| = -|B|$

   Example: compare determinants of $A = \begin{pmatrix} 1 & 5 & 6 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 1 & 5 & 6 \\ 0 & 1 & 2 \\ 0 & 2 & 0 \end{pmatrix}$

   Solution: $\det(A) = \begin{vmatrix} 1 & 5 & 6 \\ 0 & 2 & 0 \\ 0 & 1 & 2 \end{vmatrix} = 1\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} = 2\times 2 = 4$ and $\det(B) = 1\begin{vmatrix} 1 & 2 \\ 2 & 0 \end{vmatrix} = -4$

   Therefore $\det(A) = -\det(B)$

3. Adding constant multiple of a row or column of a square matrix A on to an-other row or column of A doesn't change its determinant.

   i.e. $A \xrightarrow{R_i \to k\,R_j + R_i} B, for\ i \neq j\ and\ \ k \in R, then\ \ |A| = |B|$

   Example : Compare $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 2 \end{pmatrix}$, k= $3\ and\ B = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1+3 & 2+6 & 2+9 \end{pmatrix}$

   Solution : $\det(A) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 2 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 2 & 2 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 1 & 2 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = -2 - 4 + 9 = 3$

   $$\det(B) = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1+3 & 2+6 & 2+9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 2+6 & 2+9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 1+3 & 2+9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 1+3 & 2+6 \end{vmatrix}$$

   $$= 5(2+9) - 6(2+6) - 2[4(2+9) - 6(1+3)] + 3[4(2+6) - 5(1+3)]$$

$$= 5 \times 11 - 6 \times 8 - 8 \times 11 + 12 \times 4 + 12 \times 8 - 15 \times 4$$
$$= 55 - 48 - 88 + 48 + 96 - 60$$
$$\det(B) = 3$$
$$\therefore \det(A) = \det(B)$$

4. Multiplying a row or column of a square matrix A by any constant k its determinant equals k times det(A).

   i.e. $A \xrightarrow{R_i \to kR_i} B, then\ |B| = k|A|$

   Example : Compare determinants of $A = \begin{pmatrix} 1 & 2 \\ 3 & 5 \end{pmatrix}$ $and\ B = \begin{pmatrix} 1 & 2 \\ 3 \times 4 & 5 \times 4 \end{pmatrix}$

Solution: $\det(A) = \begin{vmatrix} 1 & 2 \\ 3 & 5 \end{vmatrix} = 5 - 6 = -1\ and \det(B) = \begin{vmatrix} 1 & 2 \\ 3 \times 4 & 5 \times 4 \end{vmatrix} = 5 \times 4 - 2(3 \times 4) = -4$

$\therefore\ |B| = 4|A|$

5. If A is a square matrix of order n and k is a scalar then: $|kA| = k^n|A|$.

   Example:1. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $, if \det(A) = 3, then\ find\ determinant\ of\ B = \begin{pmatrix} 4a & 4b \\ 4c & 4d \end{pmatrix}$

   Solution: k = 4 and order of A = 2

   $$|B| = \begin{vmatrix} 4a & 4b \\ 4c & 4d \end{vmatrix} = 4a(4d) - 4b(4c) = 4^2(ad - bc) = 4^2 \det(A) = 16 * 3 = 48$$

   2. let $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, if \det(A) = 4, then\ find \det(B), B = \begin{pmatrix} 2a & 2b & 2c \\ 2d & 2e & 2f \\ 2g & 2h & 2i \end{pmatrix},$

   Solution $\boldsymbol{k = 2, and\ \ order\ of\ A\ is\ \ 3 \times 3\ , n = 3}$

   $$\boldsymbol{B} = \mathbf{2}\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}, = 2A \implies \det(B) = k^n \det(A) = 2^3(4) = 8 * 4 = 32$$

6. If a square matrix A has zero rows or columns then its determinant is zero.

   Example: find determinant of A= $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 4 & 5 & 6 \end{pmatrix}$

   Solution: $det(A) is\ determined\ using\ second\ row\ expansion$

   $$\det(A) = 0\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} - 0\begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} + 0\begin{vmatrix} 1 & 2 \\ 4 & 5 \end{vmatrix} = 0$$

7. If a square matrix A has identical rows or columns then its determinant is zero.

   Example: find determinant of A= $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 3 \\ 1 & 2 & 3 \end{pmatrix}$

   Solution: $\det(A) = 1\begin{vmatrix} 5 & 3 \\ 2 & 3 \end{vmatrix} - 2\begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = 15 - 6 - 2(12 - 3) + 3(8 - 5)$

   $$= 9 - 18 + 9 = 0$$
   $$\therefore \det(A) = 0$$

8. Determinant of a square matrix A and determinant of its transpose is the same.

   i.e. $|A| = |A^t|$

   Example: let $A = \begin{pmatrix} 1 & 0 & 2 \\ 4 & 2 & 0 \\ 0 & 1 & 2 \end{pmatrix}$ then find determinant of $A\ and\ A^t$

Solution: $\det(A) = 1\begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} - 4\begin{vmatrix} 0 & 2 \\ 1 & 2 \end{vmatrix} = 4 - 2 + 4(2) = 10$

$$\det(A^t) = \begin{vmatrix} 1 & 4 & 0 \\ 0 & 2 & 1 \\ 2 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} 2 & 1 \\ 0 & 2 \end{vmatrix} + 2\begin{vmatrix} 4 & 0 \\ 2 & 1 \end{vmatrix} = 4 + 2 * 4 = 10$$

9. Determinant of an identity matrix is always 1.

   i.e. $\det(I_n) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1^n = 1$

10. $let\ \ A\ and\ \ B$ be two square matrices of the same order , then $\det(AB) = \det(A)\det(B)$

    Example: let A= $\begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}$ $and\ B = \begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}$ $be\ matrices\ then\ show\ that$ $\det(AB) = \det(A)\det(B)$

Solution: AB = $\begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 5 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 16+5 & 20+10 \\ 4+2 & 5+4 \end{pmatrix} = \begin{pmatrix} 21 & 30 \\ 6 & 9 \end{pmatrix}$

$\det(AB) = \begin{vmatrix} 21 & 30 \\ 6 & 9 \end{vmatrix} = 21 * 9 - 30 * 6 = 189 - 180 = 9$

$de(A) = \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = 4 * 2 - 5 * 1 = 8 - 5 = 3\ and\ \det(B) = \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} = 4 * 2 - 5 * 1 = 8 - 5 = 3$

Then $\det(A)\det(B) = 3 * 3 = 9$

$\therefore \det(AB) = \det(A)\det(B)$

11. For any square matrix A $\det(A^m) = (\det(A))^m, for\ m \in Z^+$

    Example: le A and B are square matrices of order 3 with $|A| = 2\ and\ |B| = 5$, then find

    A. $\det(A^4)$ B. $(\det(A))^4$ C. $\det(3A^2)$

    Solution:

    A. $\det(A^4) = (det(A)^4 = 2^4 = 16$ $C.\det(3A^2) = 3^3(\det(A))^2 = 27 * 4 = 108$

12. Let A bean invertible square matrix, then $\det(A) = \frac{1}{\det(A^{-1})}$

    Proof: let A be an invertible matrix with inverse A^{-1}, then

    $AA^{-1} = I \Leftrightarrow \det(AA^{-1}) = \det(I)$

    $\Leftrightarrow \det(A)\det(A^{-1}) = 1$

    $$\boxed{\Rightarrow \det(A^{-1}) = \frac{1}{\det(A)}}$$

Example: let A and B be two invertible matrices of order 4 and $\det(A) = 3, \det(B) = 4, then$

Find a. $\det(A^{-1})$ b. $\det(A^{-1}B)$

Solution

a. $\det(A^{-1}) = \frac{1}{\det(A)} = \frac{1}{3}$ b. $\det(A^{-1}B) = \det(A^{-1})\det(B) = \frac{\det(B)}{\det(A^{-1})} = \frac{4}{\frac{1}{3}} = 12$

1.3. Inverse of a Square Matrix

i. Ad joint of a Square Matrix

Definition:
Adjoint of a square matrix A = (a_{ij}) is defined as the transpose of the matrix $\Delta = (\Delta_{ij})$where Δ_{ij} $is$the cofactor of the element a_{ij}.
Adjoint of A is denoted by Adj(A)= $(\Delta_{ij})^t$

Example: find **Ad**j(A) if A= $\begin{pmatrix} 1 & 0 & 1 \\ 2 & 3 & -1 \\ 4 & 0 & 0 \end{pmatrix}$

Solution: $\Delta_{11} = M_{11} = \begin{vmatrix} 3 & -1 \\ 0 & 0 \end{vmatrix} = 0$

$\Delta_{12} = -M_{12} = -\begin{vmatrix} 2 & -1 \\ 4 & 0 \end{vmatrix} = -4$

$\Delta_{13} = M_{13} = \begin{vmatrix} 2 & 3 \\ 4 & 0 \end{vmatrix} = -12$

$\Delta_{21} = -M_{21} = -\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix} = 0$

$\Delta_{22} = M_{22} = \begin{vmatrix} 1 & 1 \\ 4 & 0 \end{vmatrix} = -4$

$\Delta_{23} = -M_{23} = \begin{vmatrix} 1 & 0 \\ 4 & 0 \end{vmatrix} = 0$

$\Delta_{31} = M_{31} = \begin{vmatrix} 0 & 1 \\ 3 & -1 \end{vmatrix} = -3$

$\Delta_{32} = -M_{32} = \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix} = 3$

$\Delta_{33} = M_{33} = \begin{vmatrix} 1 & 0 \\ 2 & 3 \end{vmatrix} = 3$

Now $(\Delta_{ij}) = \begin{pmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} \end{pmatrix} = \begin{pmatrix} 0 & -4 & -12 \\ 0 & -4 & 0 \\ -3 & 3 & 3 \end{pmatrix}$

$\therefore Adj(A) = (\Delta_{ij})^t = \begin{pmatrix} 0 & -4 & -12 \\ 0 & -4 & 0 \\ -3 & 3 & 3 \end{pmatrix}^t = \begin{pmatrix} 0 & 0 & -3 \\ -4 & -4 & 3 \\ -12 & 0 & 3 \end{pmatrix}$

Example

1. Let A= $\begin{pmatrix} 1 & 1 & 2 \\ 3 & 4 & 3 \\ 2 & 2 & 1 \end{pmatrix}$, then find
   a. $det(A)$ $b. Adj(A)$ $c. A(Adj(A))$ $d. (Adj(A))A$

solution:

a. $\det(A) = 1\begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} - 1\begin{vmatrix} 3 & 3 \\ 2 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} = 4 - 6 - (3 - 6) + 2(6 - 8) = -3$

b. $Adj(A) = (\Delta_{ij})^t = \begin{pmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} \end{pmatrix}^t$

$\Delta_{11} = M_{11} = \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = -2$

$\Delta_{12} = -M_{12} = -\begin{vmatrix} 3 & 3 \\ 2 & 1 \end{vmatrix} = 3$

$\Delta_{13} = M_{13} = \begin{vmatrix} 3 & 4 \\ 2 & 2 \end{vmatrix} = -2$

$\Delta_{21} = -M_{21} = -\begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 3$

$\Delta_{22} = M_{22} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$

$\Delta_{23} = -M_{23} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 0$

$\Delta_{31} = M_{31} = \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = -5$

$\Delta_{32} = -M_{32} = \begin{vmatrix} 1 & 2 \\ 3 & 3 \end{vmatrix} = 3$

$\Delta_{33} = M_{33} = \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} = 1$

$\therefore Adj(A) = \begin{pmatrix} \Delta_{11} & \Delta_{12} & \Delta_{13} \\ \Delta_{21} & \Delta_{22} & \Delta_{23} \\ \Delta_{31} & \Delta_{32} & \Delta_{33} \end{pmatrix}^t = \begin{pmatrix} -2 & 3 & -2 \\ 3 & -3 & 0 \\ -5 & 3 & 1 \end{pmatrix}^t = \begin{pmatrix} -2 & 3 & -5 \\ 3 & -3 & 3 \\ -2 & 0 & 1 \end{pmatrix}$

c. $A\big(Adj(A)\big) = \begin{pmatrix} 1 & 1 & 2 \\ 3 & 4 & 3 \\ 2 & 2 & 1 \end{pmatrix}\begin{pmatrix} -2 & 3 & -5 \\ 3 & -3 & 3 \\ -2 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -2+3-4 & 3-3+0 & -5+3+2 \\ -6+12-6 & 9-12+0 & -15+12+3 \\ -4+6-2 & 6-6+0 & -10+6+1 \end{pmatrix}$

$= \begin{pmatrix} -3 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{pmatrix} = -3\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = -3I = (Adj(A))A$

$\therefore\ A\big(Adj(A)\big) = \big(Adj(A)\big)A = \det(A)\, I$

Note: For any non-singular square matrix A : $|A| = |A \times\ Adj(A)| = |Adj(A) \times A|$

- Square matrix A is said to be:
  - ✓ singular $iff\ |A| = 0$
  - ✓ non-singular $|A| \neq 0$