Lesson 3 : Inverse of a Square Matrix

> Definition:
> A square matrix A is invertible if and only if A is non- singular
> $A^{-1} = \frac{Adj(A)}{|A|}$

Examples

1. Which of the following matrices are singular or non-singular

   a. $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{pmatrix}$ b. $B = \begin{pmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \\ -4 & 2 & 6 \end{pmatrix}$
2. Find k such that the following matrices are singular

   a. $A = \begin{pmatrix} k & 6 \\ 4 & 3 \end{pmatrix}$ b. $B = \begin{pmatrix} 1 & 2 & -1 \\ -3 & 4 & k \\ -4 & 2k & 6 \end{pmatrix}$
3. Find the inverse of A= $\begin{pmatrix} 3-k & 6 \\ 2 & 4-k \end{pmatrix}$ if k = 1?
4. Find the inverse if it exist of the matrix

   a. A= $\begin{pmatrix} 1 & 3 \\ 2 & -1 \end{pmatrix}$ b. $B = \begin{pmatrix} 0 & -2 & -3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix}$
5. Find x if $\begin{vmatrix} -3 & -x \\ 3x & 4 \end{vmatrix} = 15$
6. In a given matrix A, if $det(A) = \frac{1}{4}$, then find $|A^{-1}|$

Solution:

1. a. $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{pmatrix} \Rightarrow \begin{vmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 0 & 1 & -1 \end{vmatrix} = 1\begin{vmatrix} 1 & -2 \\ 1 & -1 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ 0 & -1 \end{vmatrix} + 1\begin{vmatrix} 3 & 1 \\ 0 & 1 \end{vmatrix}$

$$= -1 + 2 + 6 + 3 = 10$$

$$\therefore \det(A) = 10 \Rightarrow \det(A) \neq 0 \text{, } A \text{ is non} - \text{singular}$$

b. $B = \begin{pmatrix} 1 & 2 & -1 \\ -3 & 4 & 5 \\ -4 & 2 & 6 \end{pmatrix} \Rightarrow \det(B) = 1\begin{vmatrix} 4 & 5 \\ 2 & 6 \end{vmatrix} - 2\begin{vmatrix} -3 & 5 \\ -4 & 6 \end{vmatrix} + (-1)\begin{vmatrix} -3 & 4 \\ -4 & 2 \end{vmatrix}$

$$= 24 - 10 - 2(-18 + 20) + (-1)(-6 + 16)$$

$$= 14 - 4 - 10 = 0$$

$\det(B) = 0 \Longrightarrow A \text{ is singular}$

2. a. $A = \begin{pmatrix} k & 6 \\ 4 & 3 \end{pmatrix}$, $if\ A\ is\ singular\ |A| = 0$

$$\Longrightarrow \begin{vmatrix} k & 6 \\ 4 & 3 \end{vmatrix} = 3k - 24 = 0 \Longrightarrow 3k = 24 \Longrightarrow k = 8$$

$$b.B=\begin{pmatrix}1 & 2 & -1\\ -3 & 4 & k\\ -4 & 2k & 6\end{pmatrix}, if\ B\ is\ singular\ \det(B)=0$$

$$\Longrightarrow\begin{vmatrix}1 & 2 & -1\\ -3 & 4 & k\\ -4 & 2k & 6\end{vmatrix}=1\begin{vmatrix}4 & k\\ 2k & 6\end{vmatrix}-2\begin{vmatrix}-3 & k\\ -4 & 6\end{vmatrix}+(-1)\begin{vmatrix}-3 & 4\\ -4 & 2k\end{vmatrix}=0\ \ldots.first\ row\ expansion$$

$$\Longleftrightarrow 24-2k^2-2(-18+4k)+(-1)(-12k+16)=0$$

$$\Longleftrightarrow 24-2k^2+36-8k+12k-16=0$$

$$\Longleftrightarrow -2k^2+4k+44=0 \Longleftrightarrow -k^2+2k+22=0$$

$$\Longrightarrow k=\frac{-b\pm\sqrt{b^2-4ac}}{2a}, a=-1, b=2, and\ c=22$$

$$\Longrightarrow k=1\pm\sqrt{23}$$

$$\therefore k=1+\sqrt{23}\ or\ k=1-\sqrt{23}$$

3. Find the inverse of A= $\begin{pmatrix}3-k & 6\\ 2 & 4-k\end{pmatrix}$ if k = 1?

$$A=\begin{pmatrix}3-1 & 6\\ 2 & 4-1\end{pmatrix}=\begin{pmatrix}2 & 6\\ 2 & 3\end{pmatrix}\ and\ A^{-1}=?$$

$\boldsymbol{cofctor}$, $\Delta_{11}=3$, $\Delta_{12}=-2,$ $\Delta_{21}=-6,$ $\Delta_{22}=2$

$$\Longrightarrow \Delta=\begin{pmatrix}3 & -2\\ -6 & 2\end{pmatrix}\ and\ Adj(A)=\begin{pmatrix}3 & -2\\ -6 & 2\end{pmatrix}^t=\begin{pmatrix}3 & -6\\ -2 & 2\end{pmatrix}$$

$$\Longrightarrow |A|=\begin{vmatrix}2 & 6\\ 2 & 3\end{vmatrix}=2*3-6*2=-6$$

$$\therefore\ A^{-1}=\frac{Adj(A)}{|A|}=\frac{\begin{pmatrix}3 & -6\\ -2 & 2\end{pmatrix}}{-6}=\begin{pmatrix}-\frac{3}{6} & \frac{6}{6}\\ \frac{2}{6} & -\frac{2}{6}\end{pmatrix}=\begin{pmatrix}-\frac{1}{2} & 1\\ \frac{1}{3} & -\frac{1}{3}\end{pmatrix}$$

4. a. A= $\begin{pmatrix}1 & 3\\ 2 & -1\end{pmatrix}$

Cofactors of A: $\Delta_{11}=-1$, $\Delta_{12}=-2,$ $\Delta_{21}=-3,$ $\Delta_{22}=1$

$$\Longrightarrow \Delta=\begin{pmatrix}-1 & -2\\ -3 & 1\end{pmatrix}\ and\ Adj(A)=\begin{pmatrix}-1 & -2\\ -3 & 1\end{pmatrix}^t=\begin{pmatrix}-1 & -3\\ -2 & 1\end{pmatrix}$$

$$\Longrightarrow |A|=\begin{vmatrix}1 & 3\\ 2 & -1\end{vmatrix}=-1-6=-7$$

$$\therefore A^{-1}=\frac{Adj(A)}{|A|}=\frac{\begin{pmatrix}-1 & -3\\ -2 & 1\end{pmatrix}}{-7}=\begin{pmatrix}\frac{1}{7} & \frac{3}{7}\\ \frac{2}{7} & -\frac{1}{7}\end{pmatrix}$$

$$b.B=\begin{pmatrix}0 & -2 & -3\\ 1 & 3 & 3\\ -1 & -2 & -2\end{pmatrix}$$

Cofactors of B

$\Delta_{11}=0,$ $\Delta_{21}=2,$ $\Delta_{31}=3$

$\Delta_{12}=-1,$ $\Delta_{22}=3,$ $\Delta_{32}=-3$

$\Delta_{13}=1,$ $\Delta_{23}=2$ $\Delta_{33}=2$

Then $\Delta=\begin{pmatrix}0 & -1 & 1\\ 2 & 3 & 2\\ 3 & -3 & 2\end{pmatrix}$ $and\ Adj(A)=\begin{pmatrix}0 & -1 & 1\\ 2 & 3 & 2\\ 3 & -3 & 2\end{pmatrix}^t=\begin{pmatrix}0 & 2 & 3\\ -1 & 3 & -3\\ 1 & 2 & 2\end{pmatrix}$

$|A|=0\begin{vmatrix}3 & 3\\ -2 & -2\end{vmatrix}-(-2)\begin{vmatrix}1 & 3\\ -1 & -2\end{vmatrix}+(-3)\begin{vmatrix}1 & 3\\ -1 & -2\end{vmatrix}=0+2-3=-1$

$$\therefore A^{-1}=\frac{Adj(A)}{|A|}=\frac{\begin{pmatrix}0 & 2 & 3\\ -1 & 3 & -3\\ 1 & 2 & 2\end{pmatrix}}{-1}=\begin{pmatrix}0 & -2 & -3\\ 1 & -3 & 3\\ -1 & -2 & -2\end{pmatrix}$$

5. $\begin{vmatrix}-3 & -x\\ 3x & 4\end{vmatrix}=15\Longrightarrow -3\times 4+3x\times x=15\Longrightarrow 3x^2-12=15$

$\Leftrightarrow 3x^2=15+12$

$\Leftrightarrow x^2=9\ \Longrightarrow x=\pm 3$

6. $det(A)=\frac{1}{4}$, then find $|A^{-1}|$

$$|A^{-1}|=\frac{1}{|A|}\Longrightarrow |A^{-1}|=\frac{1}{\frac{1}{4}}=4$$

$\therefore |A^{-1}|=4$

Note:

- Not every square matrix is invertible
- A square matrix is invertible if and only if its determinant is not zero
- For two square matrices A and B if AB exists and is invertible
  Then $(AB)^{-1}=B^{-1}A^{-1}$

1.3. Solution of System of Linear Equations Using Crammer's Rule

Crammer's Rule:

Consider the system of linear equations of two variables x and y:

$$a_1x+b_1y=c_1$$
$$a_2x+b_2y=c_2$$

Let A= $\begin{pmatrix}a_1 & b_1\\ a_2 & b\end{pmatrix}$ be a coefficient matrix then:

- $x=\frac{|A_x|}{|A|}=\frac{\begin{vmatrix}c_1 & b_1\\ c_2 & b_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}\ and\ y=\frac{|A_y|}{|A|}=\frac{\begin{vmatrix}a_1 & c_1\\ a_2 & c_2\end{vmatrix}}{\begin{vmatrix}a_1 & b_1\\ a_2 & b_2\end{vmatrix}}$….. Unique solution.
- $if\ |A|=0, then\ it\ has\ two\ possible\ solutions$
  - ✓ No solution when $|A_x|,|A_y|\neq 0$
  - ✓ Infinite solution when $|A_x|,|A_y|\ =0$

Examples: 1. solve the system $\begin{cases}2x-3y=7\\ 3x+5y=1\end{cases}$

Solution: A= $\begin{pmatrix}2 & -3\\ 3 & 5\end{pmatrix}$, $|A|=\begin{vmatrix}2 & -3\\ 3 & 5\end{vmatrix}=10+9=19$

$A_x=\begin{pmatrix}7 & -3\\ 1 & 5\end{pmatrix}, |A_x|=\begin{vmatrix}7 & -3\\ 1 & 5\end{vmatrix}=35+3=38$

$A_y=\begin{pmatrix}2 & 7\\ 3 & 1\end{pmatrix}, |A_y|=\begin{vmatrix}2 & 7\\ 3 & 1\end{vmatrix}=2-21=-19$

Then the system has unique solution since $|A| \neq 0$

$$x = \frac{|A_x|}{|A|} = \frac{38}{19} = 2 \quad and\ y = \frac{|A_y|}{|A|} = \frac{19}{19} = -1$$

Therefore $\{(x\,,y)\} = \{(2\,,-1)\}$

Consider the system of linear equations of three variables

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3$$

Hence determinant of coefficient matrix is

$$|A| = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ if } |A| \neq 0$$

$$x = \frac{|A_x|}{|A|} = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\ y = \frac{|A_y|}{|A|} = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},\ and\ z = \frac{|A_z|}{|A|} = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}},$$

Example solve the linear system $\begin{cases} -4x + 2y - 9z = 2 \\ 3x + 4y + z = 5 \\ x - 3y + 2z = 8 \end{cases}$

Solution: $A = \begin{pmatrix} -4 & 2 & -9 \\ 3 & 4 & 1 \\ 1 & -3 & 2 \end{pmatrix}, |A| = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 4 & 1 \\ 1 & -3 & 2 \end{vmatrix} = -4\begin{vmatrix} 4 & 1 \\ -3 & 2 \end{vmatrix} - 2\begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} + (-9)\begin{vmatrix} 3 & 4 \\ 1 & -3 \end{vmatrix}$

$$= -4(8+3) - 2(6-1) - 9(-9-4) = 63$$
$$|A| = 63$$

$$|A_x| = \begin{vmatrix} 2 & 2 & -9 \\ 5 & 4 & 1 \\ 8 & -3 & 2 \end{vmatrix}, |A_x| = 2(11) - 2(2) - 9(-47) = 22 - 4 + 423 = 441$$

$$|A_x| = 441$$

$$|A_y| = \begin{vmatrix} -4 & 2 & -9 \\ 3 & 5 & 1 \\ 1 & 8 & 2 \end{vmatrix} = -4(2) - 2(5) - 9(19) = -8 - 10 - 171 = -189$$

$$|A_y| = -189$$

$$|A_z| = \begin{vmatrix} -4 & 2 & 2 \\ 3 & 4 & 5 \\ 1 & -3 & 8 \end{vmatrix} = -4(47) - 2(19) + 2(-13) = -188 - 38 - 26 = -252$$

$$|A_z| = -252$$

Hence $x = \frac{|A_x|}{|A|} = \frac{441}{63} = 7,\ \ y = \frac{|A_y|}{|A|} = -\frac{189}{63} = -3\ and\ z = \frac{|A_z|}{|A|} = -\frac{252}{63} = -4$

So the solution set is$\{(x\,,y\,,z)\} = \{(7\,,-3\,,-4\,)\}$

1. Solve the system$\begin{cases} x + y - z = 1 \\ x - y + z = 2 \\ 2x + 2z = 3 \end{cases}$

Solution: $A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 2 \end{pmatrix}, |A| = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ 2 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -1 \\ 2 & 0 \end{vmatrix}$

$= 1(-2) - 1(0) - 1(2) = -4$

$|A| = -4$

$|A_x| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & 1 \\ 3 & 0 & 2 \end{vmatrix}, |A_x| = 1(-2) - 1(1) - 1(3) = -2 - 1 - 3 = -6$

$|A_x| = -6$

$|A_y| = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 2 & 3 & 2 \end{vmatrix} = 1(1) - 1(0) - (-1) = 1 - 0 + 1 = 2$

$|A_y| = 2$

$|A_z| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 2 & 0 & 3 \end{vmatrix} = 1(-3) - 1(-1) + 1(2) = -3 + 1 + 2 = 0$

$|A_z| = 0$

Hence $x = \frac{|A_x|}{|A|} = \frac{-6}{-4} = \frac{3}{2}$, $y = \frac{|A_y|}{|A|} = -\frac{2}{4} = -\frac{1}{2}$ and $z = \frac{|A_z|}{|A|} = \frac{0}{-4} = 0$

The system has unique solution $\{(x, y, z)\} = \{\left(\frac{3}{2}, -\frac{1}{2}, 0\right)\}$

2. Solve the system $\begin{cases} x + y - z = 1 \\ x - y - z = 2 \\ -2x + 2z = 3 \end{cases}$

Solution:

$A = \begin{pmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ -2 & 0 & 2 \end{pmatrix}, |A| = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & -1 \\ -2 & 0 & 2 \end{vmatrix} = 1\begin{vmatrix} -1 & -1 \\ 0 & 2 \end{vmatrix} - 1\begin{vmatrix} 1 & -1 \\ -2 & 2 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -1 \\ -2 & 0 \end{vmatrix}$

$= 1(-2) - 1(0) + 1(2) = 0$

$|A| = 0$

$|A_x| = \begin{vmatrix} 1 & 1 & -1 \\ 2 & -1 & -1 \\ 3 & 0 & 2 \end{vmatrix}, |A_x| = 1(-2) - 1(7) - 1(3) = -2 - 7 - 3 = -12$

$|A_x| = -12$

$|A_y| = \begin{vmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -2 & 3 & 2 \end{vmatrix} = 1(7) - 1(0) - (7) = 7 - 0 + 1 = 0$

$|A_y| = 0$

$|A_z| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ -2 & 0 & 3 \end{vmatrix} = 1(-3) - 1(7) + 1(-2) = -3 + 7 - 2 = 2$

$|A_z| = 2$

Hence $x = \frac{|A_x|}{|A|} = \frac{12}{0} = \nexists,\ \ y = \frac{|A_y|}{|A|} = \frac{0}{0}\ (inditerminate)\ and\ z = \frac{|A_z|}{|A|} = \frac{2}{0} = \nexists$

The system has no solution.

1.4. Application

- Polynomial interpolation
- Area of triangle in xy −plane
- Test for collinear points in xy-plane
- Two point equation of a line

1. Polynomial interpolation

a. Consider the points$(1,2), (2,5) and\ (3,36)$. Find an interpolating polynomial $p(x)\ of\ degree\ at\ most\ two. And\ estimate\ p(2)$

Solution: let $p(x) = a + bx + cx^2 be\ a\ polynomial\ of\ degree\ two$

$$p(1) = a + b + c = 2\ , p(2) = a + 2b + 4c = 5\ and\ a + 5b + 25c = 36$$

Then the system of linear equation becomes

$$\begin{cases} a + b + c = 2 \\ a + 2b + 4c = 5 \\ a + 5b + 25c = 36 \end{cases}$$

Using Crammer's rule

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 5 & 25 \end{pmatrix}, |A| = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 5 & 25 \end{vmatrix} = 1(25 - 20) - 1(25 - 4) + 1(5 - 2) = -13$$

$$|A_a| = \begin{vmatrix} 2 & 1 & 1 \\ 5 & 2 & 4 \\ 36 & 5 & 25 \end{vmatrix} = 2(50 - 20) - 1(135 - 144) + 1(25 - 72) = 22$$

$$|A_b| = \begin{vmatrix} 1 & 2 & 1 \\ 1 & 5 & 4 \\ 1 & 36 & 25 \end{vmatrix} = 1(50 - 144) - 2(25 - 4) + 1(36 - 5) = -105$$

$$|A_c| = \begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 5 \\ 1 & 5 & 36 \end{vmatrix} = 1(72 - 25) - 1(36 - 5) + 2(5 - 2) = 22$$

Hence $a = \frac{|A_a|}{|A|} = \frac{22}{-13} = -1.69\ , b = \frac{|A_b|}{|A|} = -\frac{105}{-13} = 8.07,\ \ c = \frac{|A_c|}{|A|} = \frac{22}{-13} = -1.69$

Therefore the required polynomial is $p(x) = -1.69 + 8.07x - 1.69x^2$

- $p(2) = -1.69 + 16.14 - 6.76 = 7.69$

2. Area of a Triangle in $xy - plane$

Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be three vertices of a triangle then:

- $$\text{Area}(\Delta ABC) = \begin{cases} \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \text{ if } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} > 0 \\ \frac{-1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}, \text{ if } \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} < 0 \end{cases}$$

Example: find the area of a triangle with vertices pass through the following points

a. $(-2,0), (0,2) and\ (2,0)$ b. $(1,0), (2,1) and\ (2,3)$

Solution:

a. $(-2,0), (0,2) and\ (2,0) \Longrightarrow \begin{vmatrix} -2 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = -2(2) + 0 + 1(-4) = -8 < 0$

$$A(\Delta) = -\frac{1}{2}\begin{vmatrix} -2 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & 1 \end{vmatrix} = -\frac{1}{2}(-8) = 4$$

Therefore area of a triangle is 4 sq.unit.

b. $(1,0), (2,1) and\ (2,3) \Longrightarrow \begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix} = 1(-2) + 0 + 1(4) = 2 > 0$

$$A(\Delta) = \frac{1}{2}\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ 2 & 3 & 1 \end{vmatrix} = \frac{1}{2}(2) = 1$$

Therefore area of a triangle is 1 sq. Unit.

c. Area of a triangle passing through points $A(2,1), B(x, 3x) and\ C(-1, 4) is\ 2sq.unit.$ Find x.

Solution: area$(\Delta) = 2$

Case 1 if $\begin{vmatrix} 2 & 1 & 1 \\ x & 3x & 1 \\ -1 & 4 & 1 \end{vmatrix} > 0$

$$A(\Delta) = \frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ x & 3x & 1 \\ -1 & 4 & 1 \end{vmatrix} = \frac{1}{2}(2(3x-4) - 1(x+1) + 1(4x+3x)$$

$$A(\Delta) = 2 \Longrightarrow \frac{6x-8-x-1+7x}{2} = 2 \Longleftrightarrow 12x - 9 = 4 \Longleftrightarrow x = \frac{13}{12}$$

Case2. if $\begin{vmatrix} 2 & 1 & 1 \\ x & 3x & 1 \\ -1 & 4 & 1 \end{vmatrix} < 0$, $A(\Delta) = -\frac{1}{2}\begin{vmatrix} 2 & 1 & 1 \\ x & 3x & 1 \\ -1 & 4 & 1 \end{vmatrix}$

$$A(\Delta) = 2 \Longrightarrow -\frac{12x-9}{2} = 2 \Longleftrightarrow 12x - 9 = -4 \Longrightarrow x = \frac{5}{12}$$

3. Test for Collinear Points in xy-Plane

Let $A(x_1, y_1), B(x_2, y_2)$ and $C(x_3, y_3)$ be three Collinear points in $xy-plane$ if and only if $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

Example: determine if the following points are collinear or not.

a. $(0,2), (1,1) and\ (3,-1)$ b. $(1,2),\ \ (1,1)\ and\ (3,-1)$

Solution:

a. $(0,2), (1,1) and\ (3,-1), test\ \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & -1 & 1 \end{vmatrix} = 0 - 2(-2) + 1(-4) = 4 - 4 = 0$$

$$\begin{vmatrix} 0 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & -1 & 1 \end{vmatrix} = 0, therefore\ the\ points\ are\ collinear$$

b. . $(1,2),\ \ (1,1)\ and\ (3,-1)$

$$test\ \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 3 & -1 & 1 \end{vmatrix} = 1(2) - 2(-2) + 1(-4) = 2 \neq 0$$

$\therefore\ therefore\ the\ points\ are\ not\ collinear$