Lesson 4: Real Numbers

Objective

Here are five specific objectives with action verbs related to the definition and properties of real numbers:

1. **Identify** the types of real numbers by **classifying** each number as either rational or irrational.
2. **Compare** two real numbers by **applying** the trichotomy property to determine if one number is less than, equal to, or greater than the other.
3. **Demonstrate** the use of the transitive property by **verifying** the order relationship among three real numbers and **proving** that if (a < b) and (b < c), then (a < c).
4. **Calculate** the midpoint of a given interval by **using** the formula ((a + b) / 2) to **find** a real number between two specified real numbers.
5. **Express** real number intervals in various forms by **converting** given intervals into interval notation, inequalities, and number line diagrams.

Brainstorming Questions

Here are three real-life application questions tailored to Ethiopian daily activities, which involve understanding and comparing real numbers, using the properties of real numbers and mathematical concepts like intervals, and absolute values. Each question is followed by a detailed explanation.

1. Comparing Costs of Local Produce

Question:

Imagine you are comparing the prices of two types of local produce in a market in Addis Ababa: tomatoes and onions. If the cost per kilogram of tomatoes is $\frac{3\sqrt{2}}{5}$ ETB and the cost per kilogram of onions is 2.5 ETB, determine which produce is cheaper. Use the properties of real numbers to make the comparison.

Explanation:

To compare $\frac{3\sqrt{2}}{5}$ ETB with 2.5 ETB, first convert $\frac{3\sqrt{2}}{5}$ to a decimal. The value of ($\sqrt{2}$ = 1.414). so:

$\frac{3\sqrt{2}}{5} \approx \frac{3*1.414}{5} \approx \frac{4.242}{5} \approx 0.8484$ ETB

Since 0.8484 ETB (for tomatoes) is less than 2.5 ETB (for onions), tomatoes are cheaper than onions. This comparison uses the trichotomy property, which states that for any two real numbers, one is less than, equal to, or greater than the other.

2. Determining the Midpoint for Selling Land

Question:

You are considering selling a piece of land between two locations: one at 8 kilometers and another at 15 kilometers from the city center of Addis Ababa. Determine the midpoint between these two locations to set an average price for the land.

Explanation:

To find the midpoint between 8 km and 15 km, calculate the average: Mid-point = $\frac{8+15}{2} = \frac{23}{2}$ = 11.5 klometers

Thus, the average distance is 11.5 kilometers. This method uses the concept of intervals to find a number between two given numbers.

3. Estimating Daily Water Consumption

Question:

A family uses water from two different sources in Addis Ababa: one is measured at 0.985 cubic meters and the other at 1.022 cubic meters. To estimate their total water consumption, find the absolute difference between the two measurements. This will help in understanding their water usage more effectively.

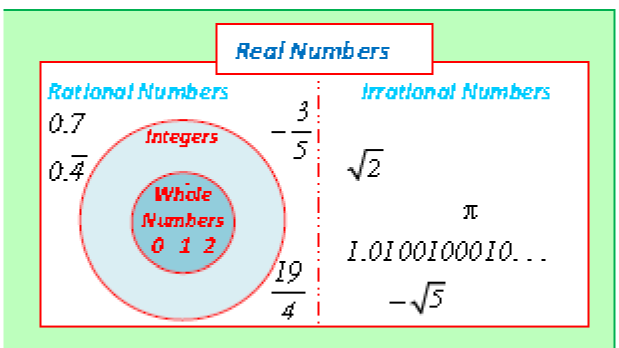
Explanation:

To find the absolute difference between 0.985 and 1.022 cubic meters: Difference = |1.022 - 0.985| = |0.037| = 0.037 cubic meters.

The absolute value is used to determine the distance between two measurements on a number line, which helps in quantifying how much more or less water was used from one source compared to the other.

Definition 1.6 Real numbers

A number is called a real number, if and only if it is either a rational number or an irrational number. The set of real numbers, denoted by R, can be described as the union of the sets of rational and irrational numbers. It can be expressed as R = { x : x is a rational number or an irrational number.} or diagrammatically,



Comparing real numbers

You have seen that there is a one to one correspondence between a point on the number line and a real number.

i. Suppose two real numbers a and b are provided. In this scenario, one of the following statements holds true: $a < b$, $a = b$, or $a > b$. This property is known as the **trichotomy property** in mathematics.

ii. For any three real numbers a, b, and c, if $a < b$ and $b < c$, then it follows that $a < c$. This property is known as the **transitive property** of order in mathematics.

Applying the above properties, we have

Given two non-negative real numbers a and b, if a^2 < b^2, then $a < b$.

Example: Compare each pair

a) $\frac{\sqrt{3}}{5}$, 0.41 b) $\frac{2}{\sqrt{3}}$, 1.12

Answer: a) When squaring the numbers $\frac{\sqrt{3}}{5}$ and 0.41, we obtain $\frac{3}{25}$ and 0.1681 respectively. Dividing 3 by 25 results in 0.12. Comparing 0.12 and 0.1681 reveals that 0.12 is less than 0.1681. Therefore, it can be concluded that $\frac{\sqrt{3}}{5}$ is less than 0.41.

Answer: b)When squaring the numbers $\frac{2}{\sqrt{3}}$ and 1.12, we obtain $\frac{4}{3}$ and 1.2544 respectively. Dividing 4 by 3 results in 1.333..... Comparing 1.33.... and 1.2544 reveals that 1.333... is greater than 1.2544. Therefore, it can be concluded that $\frac{2}{\sqrt{3}}$ is greater than 1.12."

Determining real numbers between two numbers

The number between a and b can be calculated as the average of a and b, which is given by the formula $\frac{(a+b)}{2}$. This formula provides the midpoint between the two numbers a and b.

Example 1: Find a real number between 2 and 6.8.

Answer: To find a real number between 2 and 6.8, we can simply take the average of the two numbers.

Average of 2 and 6.8:

$$\frac{2+6.8}{2} = \frac{8.8}{2} = 4.4$$

Therefore, a real number between 2 and 6.8 is 4.4.

Example 2: Find at least two real numbers between −0.54 and −0.765.

Answer:

To find at least two real numbers between -0.54 and -0.765, we can follow these steps:

Step 1: Find the average of the two given numbers: -0.54 + (-0.765) = -1.305 $\frac{-1.305}{2}$ = - 0.6525.

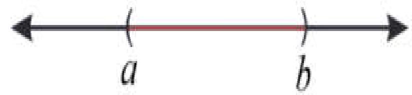
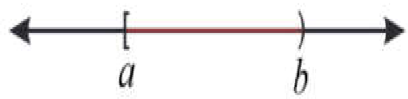
Step 2: Choose a number between the average and the lower bound: Between -0.6525 and -0.54, we can choose -0.6 as one real number.

Step 3: Choose a number between the average and the upper bound: Between - 0.6525 and - 0.765, we can choose -0.7 as another real number.

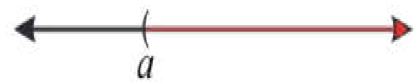
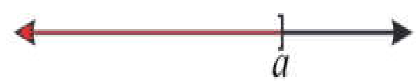
Therefore, two real numbers between - 0.54 and - 0.765 are - 0.6 and - 0.7.

Intervals

A real interval is a set that includes all real numbers lying between two specified numbers. For instance, let us consider a real number x that falls between two given real numbers a and b where.

Inequality	Interval Notation	Graph on Number Line	Description
$a < x < b$	(a,b)		x strictly between a and b
$a \le x < b$	$[a,b)$		x between a and b, including a
$a < x \le b$	$(a,b]$		x between a and b, including b
$a \le x \le b$	$[a,b]$		x between a and b, including a and b

The symbol '∞' represents infinity, signifying endlessness or the absence of an end to the right, while '–∞' denotes negative infinity, indicating endlessness or the absence of an end to the left. Real numbers within intervals can be represented using a point 'a' or 'b', along with '∞' and '–∞', as illustrated in the table below.

Inequality	Interval Notation	Graph on Number Line	Description
$x > a$	(a, ∞)		x is greater than a
$x < a$	$(-\infty, a)$		x is less than a
$x \geq a$	$[a, \infty)$		x is greater than or equal to a
$x \leq a$	$(-\infty, a]$		x is less than or equal to a

Example: Represent each of the following using a) interval notations b) inequalities c) on the number line

i) Real numbers between 3 and 5 including both end points.

ii) Real numbers between - 2 and $\frac{3}{2}$ including - 2 and excluding $\frac{3}{2}$.

iii) Real numbers on the right of 0 including 0.

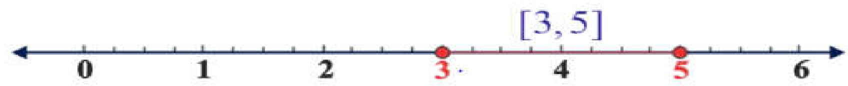
Answer:

i) Real numbers between 3 and 5 including both endpoints:

a) Interval notation: [3, 5]

b) Inequalities: $3 \leq x \leq 5$

c) Number line representation:

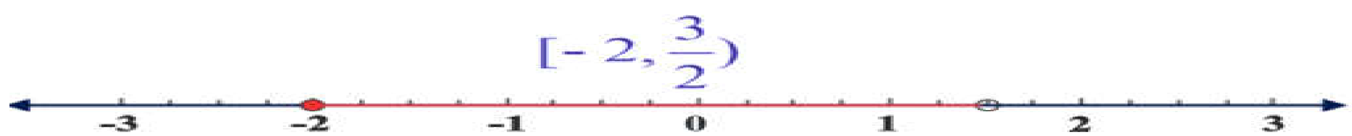


ii) Real numbers between -2 and $\frac{3}{2}$ including -2 and excluding $\frac{3}{2}$:

a) Interval notation: $[-2, \frac{3}{2})$

b) Inequalities: $-2 \leq x < \frac{3}{2}$

c) Number line representation:

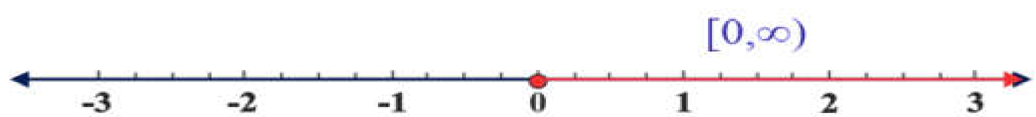


iii) Real numbers on the right of 0 including 0:

a) Interval notation: $[0, \infty)$

b) Inequalities: $x \geq 0$

c) Number line representation:



Absolute values

The absolute value of a real number, represented as $|x|$, is defined as $|x| = x$ if x is greater than or equal to 0, and as $-x$ if x is less than 0.

Example: Find the absolute value of each of the following. a) - 5 b) $\sqrt{5} - 8$

Answer: a) $|-5| = 5$

b) $|\sqrt{5} - 8| = -(\sqrt{5} - 8) = 8 - \sqrt{5}$ because $\sqrt{5} - 8 < 0$.

Therefore: $|\sqrt{5} - 8| = 8 - \sqrt{5}$

Example 2: Find the distance between - 3 and 7?

Answer: 7 - (- 3) = 7 + 3 = 10

Example 3: Determine the unknown 'x' for

a) $|x| - 5 = -3$ b) $|x - 5| = -3$ c) $|x - 5| - 5 = -3$

Answer: a) $|x| - 5 = -3$, $|x| - 5 + 5 = -3 + 5$ (add 5 to both sides),

$|x| = 2$, therefore $x = 2,\ or\ x = -2$.

b) $|x - 5| = -3$, absolute value shows the distance. Thus, it cannot be negative. Therefore, $|x - 5| = -3$ is undefined.

c) $|x - 5| - 5 = -3$, $|x - 5| - 5 + 5 = -3 + 5$ (add 5 to both sides),

$|x - 5| = 2$, Thus, $x - 5 = 2$, or $x - 5 = -2$.

$x = 2 + 5$ or $x = -2 + 5$, Therefore: $x = 7$ or $x = 3$

Exponents and radicals

If a is a real number and n is a positive integer, then

$\underbrace{a} \times a \times a \times a \times a \times \ldots \times a = =$ *a*^*n*

$n\ times$

is an exponential expression, where a is the base and n is the exponent or power.

Example: 3^4 = 3×3×3×3 = 81, 2^6 = 2×2×2×2×2×2 = 64

Note:

1. Zero exponent: If $a \neq 0$, then *a*^0 = 1.
2. Negative exponent: If $a \neq 0$ and n is positive integer, then *a*^−*n* = 1/*a*^*n*.

definition 2.10

If a^2 = b, then a is a square root of b.

If a^3 = b, then a is a cube root of b.

If n is a positive integer and a^n = b, then a is called the n^th root of b.

Note:

If b is any real number and n is a positive integer greater than 1, then the principal nth root of b is denoted by $\sqrt[n]{b}$ and is defined as the unique real number x such that x^n = b.

The expression $\sqrt[n]{b}$ is called a radical expression, where $\sqrt{\ }$ represents the radical sign, n denotes the index, and b is the radicand. When the index is not specified, the radical sign signifies the principal square root.

If $b \in \mathbb{R}$ and n is an odd positive integer greater than 1, then $b^{\frac{1}{n}} = \sqrt[n]{b}$.

If $b \geq 0$ and n is an even positive integer greater than 1, then $b^{\frac{1}{n}} = \sqrt[n]{b}$.

Law of exponents

For any a, $b \in \mathbb{R}$ and n, $m \in \mathbb{N}$, the following statement is true:

1. $a^n \times a^m = a^{n+m}$
2. $\frac{a^n}{a^m} = a^{n-m}$, where, $a \neq 0$.
3. $(a^n)^m = (a^m)^n = a^{m \times n}$
4. $(a \times b)^n = a^n \times b^n$
5. $\left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}$ for $b \neq 0$

Please note that these rules also apply to the nth power and rational power of the form $a^{\frac{n}{m}} = \left(a^{\frac{1}{m}}\right)^n$.

Please note that these rules also apply to the $(1/n)^{th}$ power and rational power of the form $\left(a^{\frac{1}{n}}\right)^m = a^{\frac{m}{n}}$

Example: Use the above rules and simplify each of the following.

a) $5^{\frac{1}{3}} \times 5^{\frac{2}{3}}$ b) $\frac{7^4}{7^2}$ c) $11^3 \times 3^3$ d) $64^{\frac{1}{3}}$

1. Answer: a) Apply the rule of multiplying powers with the same base: $a^n \times a^m = a^{n+m}$

a) $5^{\frac{1}{3}} \times 5^{\frac{2}{3}} = 5^{\frac{1+2}{3}} = 5$

b) $\frac{7^4}{7^2} = 7^{4-2} = 7^2 = 49$

c) $11^3 \times 3^3 = (11 \times 3)^3 = (33)^3$

d) $64^{\frac{1}{3}} = (2 \times 2 \times 2 \times 2 \times 2 \times 2)^{\frac{1}{3}} = (4 \times 4 \times 4)^{\frac{1}{3}} = 4$

Addition and subtraction of radicals

Radicals that have the same index and the same radicand are said to be like radicals.

Example: $-5\sqrt{7},\ 3\sqrt{7}, \frac{-4}{11}\sqrt{7}, 0.75\sqrt{7}$ are like radicals and $\sqrt{5}, 3\sqrt{7}, \sqrt[3]{7}, \sqrt[4]{7}$, …. are all unlike radicals

Example 2: Simplify $8\sqrt{27} - 11\sqrt{48} + 7\sqrt{75}$

Answer: $8\sqrt{27} - 11\sqrt{48} + 7\sqrt{75}$

$8\sqrt{9 * 3} - 11\sqrt{16 * 3} + 7\sqrt{25 * 3}$

$8(3\sqrt{3} - 11(4\sqrt{3}) + 7(5\sqrt{3})$

$24\sqrt{3} - 44\sqrt{3} + 35\sqrt{3}$

$59\sqrt{3} - 44\sqrt{3}$

$15\sqrt{3}$

Limit of accuracy

Measurements are crucial in daily life for tasks like taking a child's temperature, estimating time, measuring medicine, and determining weights, areas, and volumes. Sometimes, exact values may not be possible, leading to the need for approximate values.

In this subtopic, you will learn mathematical concepts related to approximation, including rounding numbers, significant figures (s.f.), decimal place (d.p.), and accuracy.

Rounding

Rounding off is a form of estimation used in daily life, Mathematics, and Physics. Physical quantities such as money amounts, distances, and lengths are estimated by rounding the actual number to the nearest whole number or specific decimal places.

When rounding whole numbers, identify the "rounding digit." For rounding to the nearest 10, the rounding digit is the second number from the right (tens place). For rounding to the nearest hundred, the third number from the right (hundreds place) is the rounding digit. To round off:

First, identify your rounding digit, then locate the digit to its right.

i. If the digit is 0, 1, 2, 3, or 4, keep the rounding digit the same. Any digits to the right of the rounding digit become zero.

ii. If the digit is 5, 6, 7, 8, or 9, the rounding digit increases by one. All digits to the right of the rounding digit become zero.

Example: 67,612 people live in a town. Round this number to various levels of accuracy. 38,721

a. Nearest 100: The number rounds up to 67,600, as the rounding digit is 6. The digit to the

right is 1, so the rounding digit remains the same and all digits to the right become zero.
b. Nearest 1,000: The number rounds up to 68,000. The rounding digit is 7, and the digit to the right is 6, causing the rounding digit to increase by one to 8, with all following digits becoming zero.
c. Nearest 10,000: The number rounds up to 70,000. The rounding digit is 6, and the digit to the right is 7, leading the rounding digit to increase by one to 7, with all following digits becoming zero.

In this scenario, it is unlikely that the exact number will be provided. Instead, the result is less precise but more user-friendly.

Decimal place

A number can be approximated to a specific number of decimal places (d.p.), which indicates the figures written after a decimal point.

Note that: Approximating a number to 1 decimal place involves rounding to the nearest tenth, while approximating to 2 decimal places means rounding to the nearest hundredth.

Example: Write 32.746 to a) 1 d.p b) 2 d.p c) 3 d.p d) 4 d.p

Answer: a) To round 32.746 to 1 decimal place, we look at the digit in the second decimal place, which is 4. Since 4 is less than 5, we do not need to round up. Therefore, 32.746 rounded to 1 decimal place is 32.7.

b) To round 32.746 to two decimal places, we look at the third decimal place, which is 6. Since 6 is equal to or greater than 5, we round up the second decimal place. Therefore, 32.746 rounded to two decimal places is 32.75.

c) o round 32.746 to 3 decimal places, we look at the digit in the fourth decimal place, which is 6. Since 6 is equal to or greater than 5, we round up the last digit in the third decimal place. Therefore, 32.746 rounded to 3 decimal places is 32.746.

d) To write 32.746 to 4 decimal places, we look at the digit in the 4th decimal place, which is not stated. Since the digit to the right of 6 is 0, we do not need to round up. Therefore, 32.746 to 4 decimal places is 32.7460.

Significant figures

Numbers can be approximated to a specific number of significant figures (s.f.). In the number 73.298, the 7 is the most significant figure with a value of 70, while the 8 is the least significant with a value of 8 thousandths. To show the accuracy of approximation using significant figures, count the digits from the first non-zero digit from left to right. This count represents the number of significant figures.

Example: a. Write 80.573 to 3 s.f. b. Write 0.0084 to 1 s.f. c. Write 7038 to 1 s.f.

Answer: a) First, 80.547 has five significant figures. To write the number in 3 s.f., we observe the fourth significant figure, which is 7. Since this number is not below 5, we round the third significant figure, 5, to 6. Therefore, the 3 s.f. form of 80.573 is 80.6.

b) n this case, the significant digits are 8 and 4. The number 8 is the most significant digit, and we will decide whether to round up 4. Since 4 is less than 5, we write 0.0084 as 0.008 to 1 significant figure.

c) When rounding 7038 to 1 significant figure, the result would be 7000.

Accuracy

Dear learners: In this lesson, you will learn how to approximate upper and lower bounds for data with a specified accuracy, such as rounding numbers or expressing them to a specific number of significant figures.

Simply to find upper bound add zero at the last and add 5 from the last added zero, and to find the lower bound add zero at the last and subtract 5 at the last add zeros.

Example: find the upper and lower bound for a) 3.2 b) 3.20 c) 3.200

Answer: the upper bound for 3.2 is add 0.05, 3.2 + 0.05 = 3.25 and the lower bound for 3.2 is subtract 0.05, 3.2 - 0.05 = 3.15

Answer b) the upper bound for 3.20 is add 0.005, 3.2 + 0.005 = 3.205 and the lower bound for 3.20 is subtract 0.005, 3.20 - 0.005 = 3.105

Answer c) the upper bound for 3.200 is add 0.0005, 3.200 + 0.0005 = 3.2005 and the lower bound for 3.200 is subtract 0.0005, 3.20 - 0.005 = 3.1005

Effect of operation on accuracy

For addition and multiplication two numbers

i. To find upper bounds: - if it's addition: add the two numbers upper bounds (if it's multiplication: multiply the two numbers upper bounds).

Example: upper bound for

a) 3.2 + 4.5 is 3.25 + 4.55 = 7.80

b) 3.2 x 4.5 is 3.25 x 4.55 = 14.7875

ii. To find lower bounds: - add the two lower bound numbers (if it's multiplication: multiply the two lower bound numbers)

Example: upper bound for a) 3.2 + 4.5 is 3.15 + 4.45 = 7.60

b) 3.2 x 4.5 is 3.15 x 4.45 = 14.0175

For subtraction and division two numbers

i. To find upper bounds: - subtract the lower bound of the second number from the upper bound of the first number (if it's division: divide the upper bound of the numerator to the lower bound of the denominator).

Example: upper bound for a) 9.2 - 4.5

b) $\frac{9.2}{4.5}$ is

Answer: Upper bound for a) 9.2 - 4.5 is 9.25 - 4.45 = 4.80

b) Upper bound for $\frac{9.2}{4.5}$ is

1. Find the upper bound for $\frac{9.2}{4.5}$, $\frac{9.2}{4.5} = \frac{9.25}{4.45}$

Upper bound = 2.08

Therefore, the upper bound for $\frac{9.2}{4.5}$ is approximately 2.08.

ii. To find lower bounds: - Subtraction: Subtract the upper bound second number from the lower bound of the first numbers (or if it's division: divide the lower bound numerator to upper bound of denominator).

Example: lower bound for a) 9.2 - 4.5

b) $\frac{9.2}{4.5}$ is

Answer:

a) 9.2 - 4.5 is 9.15 - 4.55 = 4.60
b) Lower bound for $\frac{9.2}{4.5}$ is $\frac{9.15}{4.55}$ = 2.056

Standard notation (Scientific notation)

Dear learners: By utilizing scientific notation and concise language, we can effectively convey the magnitudes of these numbers in a clear and understandable manner.

Large and small numbers can be tricky to work with or write out. That's why we use scientific notation, also known as standard notation, to simplify them. This format makes it easier to handle these numbers by expressing them as a coefficient multiplied by 10 raised to a certain power. This method helps maintain accuracy and clarity when dealing with extremely large or small values in scientific and mathematical contexts

Definition:

A number is said to be in scientific notation (or standard notation) if it is written as a product of the form $a \times 10$^n where $1 \leq a < 10$ and n is an integer (students text book).

Example 1: Express each of the following in standard notation.

a) 0.000000234 b) 567,200,000,000

Answer: a) 0.000000234 = 2.34 x 10^(-7) (the decimal points are moved 7 times to right to write $1 \leq d < 10$ and $d \times 10^n$, where $n \in \mathbb{Z}$.

b) 567,200,000,000 = 5.672 x 10^11 (the decimal points are moved 11 times to left to write $1 \leq d < 10$ and $d \times 10^n$, where $n \in \mathbb{Z}$..

Example 1: Write each of the following in ordinary decimal notation.

a) 2.34 x 10^(-7) b) 5.672 x 10^11

Answer: a) 2.34 x 10^(-7) = 00000000002.34 x 10^(-7), then move the decimal points to left 7 times, it will give as 0.000000234.

b) 5.672 x 10^11 = 5.67200000000 x 10^11, then move the decimal points to right 11 times, it will give as 567,200,000,000.

Rationalization

When dealing with ratios involving irrational denominators, calculating the quotient can be challenging. To simplify the process, it is often beneficial to convert the irrational denominator into a rational one. This conversion helps in making the calculations more manageable and facilitates a clearer understanding of the ratio.

The number used to multiply and rationalize the denominator is called the rationalizing factor, which is equivalent to 1.

No	Given number	Rationalizing factor
1	$\frac{1}{\sqrt{a}-\sqrt{b}}$	$\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}+\sqrt{b}}$
2	$\frac{1}{a+\sqrt{b}}$	$\frac{a-\sqrt{b}}{a-\sqrt{b}}$
3	$\frac{1}{\sqrt{a}-b}$	$\frac{\sqrt{a}+b}{\sqrt{a}+b}$
4	$\frac{1}{\sqrt{a}+\sqrt{b}}$	$\frac{\sqrt{a}-\sqrt{b}}{\sqrt{a}-\sqrt{b}}$

Example: Rationalize the denominator for each of the following.

1. $\frac{2}{\sqrt{5}}$ b) $\frac{4}{\sqrt[4]{2}}$ c) $\frac{3}{3-\sqrt{11}}$

Solution: a) the rationalizing factor for $\frac{2}{\sqrt{5}}$ is $\frac{\sqrt{5}}{\sqrt{5}}$. Thus, to rationalize the denominator of $\frac{2}{\sqrt{5}}$ we multiply it by the rationalizing factor $\frac{\sqrt{5}}{\sqrt{5}}$.. Therefore, $\frac{2}{\sqrt{5}} = \frac{2}{\sqrt{5}} \times \frac{\sqrt{5}}{\sqrt{5}}. = \frac{2\sqrt{5}}{\sqrt{5}\sqrt{5}}.= \frac{2\sqrt{5}}{5}$.

b) First the rationalizing factor of $\frac{4}{\sqrt[4]{2}}$ is $\frac{\sqrt[4]{2^3}}{\sqrt[4]{2^3}}$. Thus, $\frac{4}{\sqrt[4]{2}} = \frac{4}{\sqrt[4]{2}} \times \frac{\sqrt[4]{2^3}}{\sqrt[4]{2^3}} = \frac{4\times\sqrt[4]{2^3}}{\sqrt[4]{2}\times\sqrt[4]{2^3}} = \frac{4\sqrt[4]{8}}{\sqrt[4]{2^{1+3}}} = \frac{4\sqrt[4]{8}}{2}$

c) The rationalizing factor for $\frac{3}{3-\sqrt{11}}$ is $\frac{3+\sqrt{11}}{3+\sqrt{11}}$. Therefore, Step 1: Multiply by the rationalizing factor. The rationalizing factor of $\frac{3}{3-\sqrt{11}} \times \frac{3+\sqrt{11}}{3+\sqrt{11}}$..

$$\frac{3\left(3+\sqrt{11}\right)}{\left(3-\sqrt{11}\right)\left(3+\sqrt{11}\right)} = \frac{9+3\sqrt{11}}{9-11} = \frac{9+3\sqrt{11}}{-2}$$